o

SECONDARY-SCHOOL MATHEMATICS

BY

ROBERT L. SHORT
HEAD OF THE DEPARTMENT OF MATHEMATICS
CLEVELAND TECHNICAL HIGH SCHOOL

AND

WILLIAM H. ELSON
SUPERINTENDENT OF SCHOOLS, CLEVELAND, OHIO

BOOK II

D. C. HEATH & CO., PUBLISHERS
BOSTON NEW YORK CHICAGO

2 B 0

PREFACE

This volume continues the work of Book I and completes the required work in most four-year high schools; namely, algebra through quadratic equations, and all of plane geometry. In addition it trains the pupil in the use of logarithms and trigonometry of the right triangle. These last two topics are taught as a direct consequence of the theory of exponents and of ratios in similar right triangles.

With algebra through proportion and most of straight line geometry at the student's command, this book offers from the very beginning the greatest opportunities for the introduction of interesting problems which are impossible in the old plan of keeping algebra and geometry separate.

The applied problems, involving a knowledge of technical terms in shop or in the work of household arts, are so arranged that their introduction is at the discretion of the teacher, this of course depending largely on the type of school in which the text is being used.

Teachers and pupils alike will enjoy the unique combinations of algebra, geometry, and trigonometry under circle theorems.

Considerable stress is placed on the uses of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, as these are of constant occurrence in solving squares, triangles, pentagons, and hexagons.

As in Book I, we are indebted to Miss Carlotta Greer, Cleveland Technical High School, and Professor Kenneth G. Smith, University of Wisconsin, for much of the work in Applied Mathematics. We also thank those who have been helpful in criticising the manuscripts and in eliminating errors from the proof.

THE AUTHORS.

FOR THE TEACHER

THIS volume extends the work to cover some features of higher mathematics. Beginning with page 188, considerable attention is devoted to projection. The problems are simple, but are such as require thought on the part of the pupil. Following the introduction of the graph are many problems of an analytic nature. These are new in type, and afford the pupil an introduction to elementary work in advanced subjects without his realizing that he is in a new field.

Enforce the use of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ as factors, until the final answer is reached. Teach the pupil to watch for combinations containing these factors. They are common to a majority of computation problems, especially those relating to geometric figures.

Fractional exponents and radical notation should be taught interchangeably, with emphasis on the exponent. The practice of using the exponential form simplifies many reductions, especially in monomial forms, and the reduction of exponents to the same denomination (pages 233–234) is sure to avoid confusion in placing coefficients under the radical sign, and in removing factors from a surd.

In such problems as those found on page 238, solve as many as possible orally. Many such problems reduce to the form $s\sqrt{2}$, where s is the side of a square, and most of the diagonals $(s\sqrt{2})$ are small numbers after substitution is effected.

Teachers will find pleasure in presenting the method of completing the square (page 250). This use of factors makes possible the reduction of the first member of every quadratic equation to the form $a^2 + 2ab$ without any actual

multiplication except in the second member of the equation. For example, $7x^2 - 12x = 64$ immediately becomes $(14x)^2 - 2(14x)12 = 64 \cdot 28$, in which the only multiplication ever necessary is the multiplication of the absolute term by four times the coefficient of x^2. Even this may sometimes be avoided, *e.g.*,

$$\begin{aligned}(14x)^2 - 2(14x)12 + 12^2 &= 144 + 64 \cdot 28\\ &= 16(9 + 4 \cdot 28)\\ &= 16 \cdot 121.\\ \therefore 14x - 12 &= \pm 4 \cdot 11.\end{aligned}$$

We have found that problems of the types of those on pages 277–280 are of great interest.

The treatment of logarithms is unique and effective.

In general, only a few of the supplemental problems for boys should be used, unless some work in manual training has been given.

TABLE OF CONTENTS

BOOK II

SECONDARY-SCHOOL MATHEMATICS

BOOK II

CHAPTER X

Proportion

174. Review § 173 and exercise 60. The relation of one number to another is often expressed in fractional form. These fractions are known as **ratios**. Thus, the ratio of 2 to 3 is written $\frac{2}{3}$. This was first written $2 \div 3$, then the division sign was modified to $2:3$, now the fractional form is preferred.

A **proportion** is the equality of two ratios. It is therefore simply a fractional equation of two terms.

Thus, $\frac{a}{b} = \frac{c}{d}$ is a proportion. This is read a divided by b is equal to c divided by d.

The numerators are the **antecedents**.

The denominators are the **consequents**.

The first antecedent and the last consequent are the **extremes**.

The first consequent and the second antecedent are the **means**.

Thus, a and d are extremes and b and c the means.

Properties of Proportion

1. *The product of the extremes is equal to the product of the means.*

This is easily proved by clearing the equation (proportion),

$$\frac{a}{b} = \frac{c}{d},$$

of fractions (§ 173). Whence $ad = bc$.

2. *If the product of two numbers is equal to the product of two other numbers, a proportion may be formed, making one product the means and the other product the extremes.*

Thus, $$xy = mc.$$

Dividing by $y \cdot m$, $$\frac{x}{m} = \frac{c}{y}.$$

Had you divided by $y \cdot c$, the proportion would have been

$$\frac{x}{c} = \frac{m}{y}.$$ (See 6, § 174.)

3. *If four quantities are in proportion, they are in proportion by* **composition.**

Let $$\frac{a}{b} = \frac{c}{d}.$$

Then, $$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or, $$\frac{a+b}{b} = \frac{c+d}{d}.$$

4. *If four quantities are in proportion, they are in proportion by* **division.**

Let $$\frac{a}{b} = \frac{c}{d}.$$

Then, $$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

or, $$\frac{a-b}{b} = \frac{c-d}{d}.$$

5. *If four quantities are in proportion, they are in proportion by* **composition** *and* **division.**

Let $$\frac{a}{b} = \frac{c}{d}. \quad (1)$$

By 3, $$\frac{a+b}{b} = \frac{c+d}{d}. \quad (2)$$

By 4, $$\frac{a-b}{b} = \frac{c-d}{d}. \quad (3)$$

Dividing (2) by (3), $$\frac{a+b}{a-b} = \frac{c+d}{c-d}. \quad (4)$$

6. *If four quantities are in proportion, they are in proportion by* **alternation.**

Thus, if $\frac{a}{b}=\frac{c}{d}$ then $\frac{a}{c}=\frac{b}{d}$.

175. In a **mean proportion** the means are equal.

Thus, $\frac{a}{x}=\frac{x}{d}$ is a mean proportion.

Solving, $x=\sqrt{ad}$.

That is, a mean proportional between a and d is the square root of their product.

The last consequent of a mean proportion is a *third proportional* to the other two numbers.

Thus, in $\frac{a}{x}=\frac{x}{d}$, d is a third proportional to a and x.

The **fourth proportional** is the last consequent in such a proportion as $\frac{a}{b}=\frac{c}{d}$.

EXERCISE 61

1. Find a mean proportional between 9 and 16.
2. Find a mean proportional between 9 and 25.
3. Find a mean proportional between 4 and 25.
4. Find a mean proportional between 4 and 36.
5. Find a mean proportional between 289 and 256.

$$\frac{289}{x}=\frac{x}{256}. \quad \text{But } \sqrt{ab}=\sqrt{a}\sqrt{b}.$$

Then, $$x^2=289\cdot 256,$$

or, $$x=\sqrt{289\cdot 256}=\sqrt{289}\sqrt{256}$$
$$=17\cdot 16=272.$$

6. Find a mean proportional between 144 and 361.
7. Is -12 a mean proportional between 4 and 36?
8. Two lines are 81′ and 121′, respectively. Find a line equal to their mean proportional.

9. Find a mean proportional between 4.41 and 2.56.

10. Find a mean proportional between .49 and 1.96.

11. Find a fourth proportional to 2, 3, 8.

12. Find a fourth proportional to 1, 5, 7.

13. Find a fourth proportional to 5, $7\frac{1}{2}$, 9.

Note that in such proportions as $\frac{2}{3} = \frac{6}{x}$, x may be obtained by inspection. For since $6 = 3 \cdot 2$ (the first numerator), x must be $3 \cdot 3$ (the first denominator). Or, reading vertically instead of horizontally, since the first consequent is $1\frac{1}{2}$ times the first antecedent, the second consequent must be $1\frac{1}{2}$ times the second antecedent.

14. Find a third proportional to 2, 6.

15. Find a third proportional to 3, 5.

16. Find a third proportional to 3, 17.

17. Find a third proportional to 2, 21.

18. Find a mean proportional between 2.21 and 1.44.

19. Find a mean proportional between .0196 and 19,600.

20. Find a mean proportional between .0225 and 2.25.

21. Find a mean proportional between $(x+3)(x+5)$ and $x^2 + 8x + 15$.

22. Find a mean proportional between $\frac{x^2 - 6x + 8}{x^2 + 4x - 12}$ and $\frac{x^2 - 6x - 72}{x^2 - 16x + 48}$.

23. Find a mean proportional between $\frac{x^2 - 3x - 10}{2x^2 - 7x - 15}$ and $\frac{2x^2 + 21x + 27}{x^2 + 11x + 18}$.

24. Find a fourth proportional to $x^2 - 13x + 12$, $x^2 - 5x - 84$, and $x^2 + 6x - 7$.

25. Find a fourth proportional to $6x^2 - x - 40$, $15x^2 - 16x - 64$, and $20x^2 + 46x - 10$.

26. In a semicircle, if a perpendicular is dropped to the *diameter, from a point* in the circumference, the perpendicular

is a mean proportional between the segments of the diameter. The diameter of the circle is 34″, and the perpendicular to it from the circumference is 15″. Find the segments of the diameter.

27. The segments AB and BC of a diameter AC are 1.44 and 3.24, respectively. Find the length of the perpendicular to AC erected at B and extending to the circumference.

28. $\frac{2x+8}{2x-3}=\frac{5x+11}{5x-11}$. Solve, using § 174, 5, before clearing of fractions.

29. $\begin{cases} \frac{(x+7)+(3y-1)}{(x+7)-(3y-1)}=\frac{7}{2}. \\ 5x-7y=-9. \end{cases}$

(Use § 174, 5, on the first equation before clearing it of fractions.)

30. Solve: $\frac{4x+8}{4x-2}=\frac{3x+10}{3x-10}$.

31. $\frac{8x+7}{8x+1}=\frac{5x+15}{15-3x}$.

32. $\frac{x-1}{x+3}=\frac{x-5}{x-3}$.

Ratio plays a very important part in science, though the ratio idea is often disguised to such an extent by the scientific notation that the pupil thinks in other terms than those of ratio or measurement.

For example, the mysteries of *Specific Gravity* disappear when one *feels* that the specific gravity is simply the ratio of the weight of a volume of some substance to that of an equal volume of some other substance taken as a standard.

The standard for liquids and solids is water.

1 cubic centimeter (c.c.) of water weighs 1 gram, or 1 cubic foot weighs $62\frac{1}{2}$ pounds.

For gases the standard is usually hydrogen. Sometimes air, which is 14.44 times as heavy as hydrogen, is used.

Ex. A cubic foot of steel weighs 490 pounds. Find specific gravity of steel.

Specific gravity of steel $= \frac{490}{62.5} = 7.84.$

It is customary to write specific gravity in a decimal form, not as a common fraction.

Units to be remembered

1″ = 2.54 centimeters.
1 liter = 1000 cubic centimeters (c.c.).
1 kilogram = 1000 grams.
1 c.c. water weighs 1 gram.
1 liter hydrogen weighs 0.09 gram.
Specific gravity air (hydrogen standard) is 14.44.

33. Ice weighs 57.5 pounds to the cubic foot. Find its specific gravity.

34. The specific gravity of oak is 0.8. Find the weight of 1 cubic foot.

35. A cubic foot of lead weighs 706 pounds. Find its specific gravity.

36. A cubic foot of copper weighs 550 pounds. Find its specific gravity.

37. The specific gravity of aluminum is 2.6. Steel is how many times as heavy?

38. A liter of hydrogen weighs 0.09 gram. How many liters in 12 grams of hydrogen?

39. A gas is 1.03 times as heavy as air. What is its specific gravity (hydrogen standard)? What is its specific gravity (air standard)?

40. Nine grams of hydrogen have a volume of 100 liters. Find the weight of 17 liters. Find the volume of 3.5 grams.

41. The specific gravity of a certain gas is 8.5 (hydrogen standard). Find the weight of 1 liter of it.

42. The specific gravity of a certain gas is 35.5. Find its specific gravity (air standard).

43. The specific gravity of sea water is 1.024. What is the *weight of 5 gallons?*

44. The volume of a gas is inversely proportional to the pressure upon it. That is, if under a certain pressure p_1, a gas has a volume v_1, and at another pressure p_2, the gas has a volume v_2, this law may be stated in the proportion:

$$\frac{v_1}{v_2} = \frac{p_2}{p_1}.$$

100 cu. ft. of gas under a pressure of 5 pounds has what volume when the pressure is 7 pounds? 2 pounds?

45. A gas has a volume of 90 cu. ft. when the pressure is 6 pounds. What pressure will reduce the volume to 60 cu. ft.?

46. The specific gravity of copper is 8.9. A piece of copper is drawn out into a wire $\frac{1}{4}''$ in diameter. This coil of wire weighs 552 pounds. What is the length of the wire?

176. Proportion is of value in the study of similar figures in geometry.

Similar figures are polygons which are mutually equiangular and whose corresponding sides are in proportion.

Problem

177. *To divide a line into any number of equal parts.*

1. Take AB the given line.

2. From A draw an indefinite line AK making any convenient angle with AB.

3. On AK take equal parts AX, XY, YZ.

4. Connect Z and B.

5. Draw YM and $XO \parallel ZB$, meeting AB at M and O, respectively.

6. Then $AO = OM = MB$ (§ 137) and AB is trisected.

7. Proceed in a similar manner for any required number of equal parts of a line, laying off n equal parts on AK.

EXERCISE 62

1. Draw a line 10″ long. Divide it into three equal parts; into four equal parts.

2. Separate a given line XY into five equal parts.

3. Draw a line 7 inches long and divide it into eight equal parts.

4. Divide a five-inch line into six equal parts.

178. Two lines are divided proportionally when their corresponding segments are in proportion. Thus, if line AB is divided at K and line CD at F, so that $\frac{AK}{CF}=\frac{KB}{FD}=\frac{AB}{CD}$, AB and CD are divided proportionally.

Theorem XXXV

179. *A parallel to one side of a triangle divides the other sides proportionally.*

Draw $\triangle ABC$. Draw $DE \parallel AB$, meeting AC and BC at D and E, respectively.

We then have

Given $\triangle ABC$ and $DE \parallel AB$, and segments AD, BE, DC, EC.

To prove $$\frac{CD}{DA}=\frac{CE}{EB}.$$

Proof. 1. Apply some unit of measure to CD and DA, small enough to be contained an integral number of times in each segment.

2. At each point of division of AC draw a parallel to AB.

3. CB will then be divided into equal parts. (§ 137.)

4. Suppose this unit of measure is contained 5 times in CD and 3 times in DA.

5. Then find the ratio of CE to EB.

6. Then compare the ratios $\frac{CD}{DA}$ and $\frac{CE}{EB}$. The conclusions *are left to the pupil.*

EXERCISE 63

1. If $\frac{CD}{DA}=\frac{CE}{EB}$ in § 179, show that $\frac{CA}{DA}=\frac{CB}{EB}$. (§ 174, 3.)

2. If $\frac{a}{b}=\frac{c}{d}$, the proportion may be written $\frac{b}{a}=\frac{d}{c}$. (§ 174, 2.)

3. If $\frac{CD}{DA}=\frac{CE}{EB}$, show that $\frac{CA}{CD}=\frac{CB}{CE}$. (§ 179.)

4. In the triangle of § 179, if $CD=8$, $CA=11$, $CE=12$, find EB.

THEOREM XXXVI

180. *In any triangle the bisector of an angle divides the opposite side into segments proportional to the sides forming the angle which was bisected.*

Draw $\triangle ABC$ and CE bisecting $\angle C$ meeting AB at E.

We now have

Given $\triangle ABC$ and CE bisecting $\angle C$.

To prove $$\frac{AE}{EB}=\frac{CA}{CB}.$$

Proof. 1. Draw $AD \parallel EC$, meeting BC produced at D. Call $\angle DAC$, o; $\angle ACE$, y; $\angle ECB$, x.

2. $x=\angle D$. (?)
3. $y=x$. (?)
4. $o=y$. (?)
5. From 2, 3, 4, $o=\angle D$. (?)
6. Then $DC=CA$. (§ 128.)
7. In $\triangle ABD$, $\frac{AE}{EB}=\frac{DC}{CB}$.
8. Substituting CA for its equal DC, we have from 7,

$$\frac{AE}{EB}=\frac{CA}{CB}.$$

Theorem XXXVII

181. *In any triangle the bisector of an exterior angle divides the opposite side produced into segments proportional to the sides forming the angle bisected.*

(This is called dividing a line externally. When a line is divided, the segments extend from the point of division to the ends of the line.)

Draw $\triangle ABC$, produce BC to H, and bisect $\angle ACH$. Let this bisector meet BA produced at K. We now have

Given $\triangle ABC$ with exterior $\angle ACH$ bisected by CK.

To prove $\dfrac{KA}{KB}=\dfrac{AC}{CB}$.

Proof. 1. Draw $AG \parallel KC$, meeting CB at G. Call $\angle HCK$, z; $\angle KCA$, x; $\angle CAG$, y; $\angle CGA$, o.

2. Since $AG \parallel KC$, in $\triangle KBC$,

$$\frac{KA}{KB}=\frac{CG}{CB}. \quad \text{(Exercise 63, 3.)}$$

3. $z=x$. (Hyp.)

4. $x=y$. (?)

5. $z=o$. (?)

6. $\therefore y=o$. (?)

7. Then, $AC=CG$. (?)

8. Substituting AC for CG in 2, we have ?.

EXERCISE 64

1. In a triangle ABC, $AB=16$, $BC=14$, $CA=13$. A parallel to AB divides CA in the ratio $\frac{8}{5}$. Find the segments into which it divides BC.

2. In a $\triangle ABC$, $XY \parallel AB$ divides CA into segments of $8''$ and $10''$, respectively. $CY=9''$. Find CB.

3. In $\triangle ABC$, DE is drawn parallel to AB. This forms two similar triangles, CDE and ABC. Why? When are two *triangles similar?* (§ 176.)

4. In $\triangle$ ABC, DE is parallel to AB. $DA = 4\frac{1}{2}$, $CA = 15$, $DE = 12$. Find AB.

Hint. Through E draw a parallel to CA.

5. In example 4, if $CE=4.3$, $EB=9.3$, $DA=12.9$, find CA.

6. In $\triangle$ ABC, $AB = 12$, $BC = 16$, $CA = 14$. The $\angle$ C is bisected. Find the segments of AB formed by the bisector.

7. The sides of a triangle are 1.2″, 2.2″, 3.5″. Find the segments of 1.2 formed by the bisector of the opposite angle.

8. The sides of a triangle are $AB = 8$, $BC = 24$, $CA = 18$. The exterior angle at C is bisected. Find the segments of AB formed by this bisector.

9. The bisector of an angle of a triangle divides the base in the ratio $\frac{5}{7}$. One side of a triangle is 15. Find the other side.

10. Three parallels cut a line into segments whose ratio is $\frac{2}{3}$. What ratio will the segments of a perpendicular to one of these parallels have? If the first and second parallels are 8″ apart, what is the distance between the first and third parallels?

11. In a right triangle one angle is 60°. Find the ratio of the hypotenuse to the shorter side.

12. In example 11, find the ratio of the legs. Can you find an exact measure of this ratio?

13. Divide line AB into segments proportional to 2, 3, 4. At A draw an indefinite line AK making any convenient angle with AB. On AK take $AX = 2$, $XY = 3$, $YZ = 4$. Connect Z and B; at Y and X draw $\parallel$s to ZB. AB is then divided in the ratio of 2, 3, 4. Why?

14. Divide a line 11 inches long in the ratio $\frac{2}{5}$.

15. Divide a line 11 inches long in the ratio 1, 4, 2.

16. The base of a triangle is 18″. The sides are 13″ and 15″. Draw a line parallel to the base, terminating in the sides and 15″ in length.

17. A line parallel to the base of a triangle is 15 inches long. The segments of one side of the triangle are 12 and 16. Find the base of the triangle.

THEOREM XXXVIII

182. *Two mutually equiangular triangles are similar.*

Draw ⨺ ABC and DEF having $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$. We have

Given two mutually equiangular triangles, ABC and DEF.

To prove ⨺ ABC and DEF similar.

Proof. 1. Place $\triangle ABC$ on $\triangle DEF$, so that $\angle A$ will coincide with $\angle D$, B and C falling at B' and C', respectively, on DE and DF.

2. Then, $$\frac{DB'}{DE} = \frac{DC'}{DF}, \quad (?)$$

or, $$\frac{AB}{DE} = \frac{AC}{DF}.$$

3. By placing $\angle B$ on $\angle E$, we have,

$$\frac{AB}{DE} = \frac{BC}{EF}. \quad (?)$$

4. Hence, $$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

5. Then, ⨺ ABC and DEF having their homologous sides proportional are similar. (§ 176.)

Under what conditions are two polygons similar? Then what must you always prove in order that two polygons may be similar?

183. In similar triangles, homologous sides lie opposite *equal angles.* (Compare § 79.)

THEOREM XXXIX

184. *A line which divides two sides of a triangle proportionally is parallel to the third side.* (Compare Theorem XXXV.)

Draw $\triangle ABC$. Draw DE in such a manner that

$$\frac{CD}{CA}=\frac{CE}{CB}.$$

We have

Given $\triangle ABC$, with $\frac{CD}{CA}=\frac{CE}{CB}$.

To prove $DE \parallel AB$.

Proof. 1. Through D draw $DK \parallel AB$.

2. Then, $\frac{CD}{CA}=\frac{CK}{CB}$. (?)

3. But since $\frac{CD}{CA}=\frac{CE}{CB}$, we have $\frac{CK}{CB}=\frac{CE}{CB}$.

4. Therefore, K and E coincide, and since $DK \parallel AB$, DE is parallel to AB.

THEOREM XL

185. *Two triangles are similar if their homologous sides are proportional.*

Given two triangles ABC and DEF with $\frac{AB}{DE}=\frac{BC}{EF}=\frac{CA}{FD}$.

To prove $\triangle ABC$ similar to $\triangle DEF$.

Proof. 1. On CA take $CK = FD$; on BC take $MC = EF$. Draw KM.

2. Then, $\triangle\triangle ABC$ and KMC are similar. (§ 184.)

3. Hence, $\frac{CA}{CK}=\frac{AB}{KM}$. And since $CK = FD$, KM must equal DE.

4. Then, $\triangle KMC = \triangle DEF$.

5. And since $\triangle\triangle KMC$ and ABC are similar, $\triangle\triangle DEF$ and ABC are similar.

EXERCISE 65

1. Two triangles are similar if two angles of one are respectively equal to two angles of the other.

2. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

3. The sides of a triangle are 9, 12, 15. One side of a similar triangle is 12. Find the other sides. How many triangles can satisfy these conditions?

4. The segments of the diagonals of a trapezoid may be written as a proportion.

5. One angle of an isosceles triangle is 30°. Construct a similar triangle whose base is 17′. How many triangles satisfy these conditions?

6. A post 3 feet high casts a shadow 4 feet long. At the same time a flagpole casts a shadow 90 feet long. How high is the flagpole?

THEOREM XLI

186. *Two triangles are similar if an angle of one is equal to an angle of the other, and the sides including these angles are proportional.*

Place one triangle on the other so that the equal angles coincide, then use § 184.

THEOREM XLII

187. *Two triangles are similar when their sides are mutually perpendicular.*

Extend the sides of the given triangles until they meet. Quadrilaterals will then be formed, two of whose opposite angles will be 90° each. The other two angles of the quadrilateral will be supplementary and can be used in proving this theorem.

EXERCISE 66

1. If the sides of two triangles are mutually parallel, the triangles are similar. (Exercise 35, 5.)

2. The altitudes upon the legs of an isosceles triangle divide each other proportionally.

Theorem XLIII

188. *If two triangles are similar, homologous altitudes are in the same ratio as homologous sides.*

Draw similar triangles ABC and DEF. Draw altitudes CX and FY.

Compare triangles AXC and DYF. (The proof is left to the pupil.) (Exercise 65, 2.)

Theorem XLIV

189. *Two polygons are similar if they are composed of triangles, similar each to each, of the same number, and similarly placed.* (*Three conditions.*)

Given two polygons $ABCDE$, $FGHIK$ such that $\triangle$s ABE, BCE, CDE are respectively similar to $\triangle$s FGK, GHK, HIK.

To prove polygons DA and IF similar. (Our proof must conform to § 176.)

Proof. 1. These similar triangles have their sides proportional. The sides of the polygon are therefore proportional.

2. $\angle ABE = \angle FGK$; $\angle EBC = \angle KGH$.

3. Therefore, $\angle ABE + \angle EBC = \angle FGK + \angle KGH$. Or, $\angle ABC = \angle FGH$.

4. In like manner the remaining homologous angles of the polygons may be proved equal.

5. Hence the polygons are similar.

The converse of this proposition is also true.

State the converse.

190. In a series of equal ratios, the sum of the numerators and the sum of the denominators are in the same ratio as any ratio of the series.

For if $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}$,

1. $\frac{a}{b}=\frac{c}{d}$.

2. $\frac{a}{b}=\frac{e}{f}$.

3. $\frac{a}{b}=\frac{g}{h}$.

4. $\frac{a}{b}=\frac{a}{b}$.

Clearing each equation, 1, 2, 3, 4, of fractions, then adding the members of the four equations, we have

$$bc+be+bg+ba=ad+af+ah+ab,$$

or, $$b(c+e+g+a)=a(d+f+h+b).$$

Then, $$\frac{c+e+g+a}{d+f+h+b}=\frac{a}{b}. \quad (\S\,174,\ 2.)$$

THEOREM XLV

191. *The perimeters of two similar polygons are in the same ratio as any two homologous sides.*

Given two similar polygons whose sides are a, b, c, d, e and a', b', c', d', e', respectively, a being homologous to a'.

To prove $\frac{a+b+c+d+e}{a'+b'+c'+d'+e'}=\frac{a}{a'}=\frac{b}{b'}$, etc.

Proof. 1. $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}=\frac{d}{d'}=\frac{e}{e'}$. (By Def.)

2. Then use § 190. (The proof is left to the pupil.)

THEOREM XLVI

192. *If a perpendicular is drawn from the vertex of the right angle to the hypotenuse of a right triangle,*

I. *The triangles formed are similar to the whole triangle, and to each other.*

II. *The perpendicular is a mean proportional between the segments of the hypotenuse.**

III. *Either leg is a mean proportional between the whole hypotenuse and the adjacent segment of the hypotenuse.*

Draw right $\triangle ABC$, right-angled at C. Draw $CE \perp AB$. We have

Given rt. $\triangle ABC$ with $CE \perp AB$.

I. **To prove** $\triangle AEC$ similar to $\triangle ABC$.

Proof. 1. These are rt. $\triangle$s, each having the $\angle A$.

2. Then, $\triangle$s AEC and ABC are similar. (Exercise 65, 2.)

3. In like manner prove $\triangle EBC$ similar to $\triangle ABC$.

II. **To prove** $\overline{CE}^2 = AE \cdot EB$.

Proof. (Compare $\triangle$s AEC and EBC.)

III. **To prove** $\overline{AC}^2 = AB \cdot AE$, $\overline{BC}^2 = AB \cdot EB$.

Proof. (Compare homologous sides of $\triangle$s ACE and ABC.)

THEOREM XLVII

193. *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the legs.*

Draw rt. $\triangle ABC$, AB being the hypotenuse. Draw $CE \perp AB$. (Add the two equations obtained from § 192, III, remembering that $AE + EB = AB$.)

HISTORICAL NOTE. This is one of the famous theorems of geometry. For hundreds of years mathematicians tried to find a square equal to the sum of two squares. It is thought that as early as 2000 B.C. the

* Always read segments from the point of division to the ends of the line. This will be found *especially* helpful in Theorems XLVIII and XLIX.

Egyptian surveyors (called "rope stretchers") knew that the square on the hypotenuse was equal to the sum of the squares on the legs when the sides were in the ratio 3, 4, 5. That is, they could tie knots in a rope at intervals of 3, 4, 5 units and form a right triangle with these segments. It was not until 500 B.C. that Pythagoras proved the theorem true for any right triangle. The theorem is known as the Pythagorean theorem. Pythagoras' proof was undoubtedly a geometric one and in recent years many proofs, both algebraic and geometric, have been given. The proof above outlined is algebraic, and is the most simple one known.

194. The **projection of a point** upon a straight line is the foot of the perpendicular drawn from the point to the line. Thus, the projection of the vertex of an isosceles triangle upon the base is the middle point of the base.

The **projection of a segment of a straight line** upon a line of indefinite length is the portion of the indefinite line included between the projections of the extremities of the first line. Thus, the projection of one leg of an isosceles triangle upon the base is the one half of the base adjacent to the leg projected.

EXERCISE 67

1. In a triangle ABC, $\angle C = 90°$, $\angle A = 30°$, $AB = 16$. Find the length of the projection of AB upon BC.

2. The difference in the direction of the intersecting lines AB and CD is 30°. The distance from A to CD is 20′, and from B to CD is 14′. How long is AB?

3. One side of an isosceles triangle is 24″, the vertical angle of the triangle is 60°. Find the projection of one side upon the base.

4. Line AB intersects line CD and is inclined 60° to line CD. AB is 18″, and from B to CD is 8″. Find projection of AB upon CD.

5. The projections of two sides of a right triangle upon the hypotenuse are 9″ and 16″, respectively. Find the altitude upon the hypotenuse.

6. The hypotenuse of a right triangle is 25″. The projection of one leg upon the hypotenuse is $\frac{3}{4}'$. Find the legs.

7. One segment of a hypotenuse is 1.2. The altitude upon the hypotenuse is 1.8. Find the other lines of the triangle.

8. One angle of a right triangle is 45°. The projection of one leg upon the hypotenuse is 41. Find the altitude upon the hypotenuse.

9. The perimeters of two similar polygons are 441 and 231, respectively. One side of the first is 21. Find the corresponding side of the second.

10. Two homologous sides of two similar polygons are 54′ and 81′, respectively. Find the ratio of their perimeters.

11. A city lot is in the form of a right triangle. One side of the lot is 15 rods. The projection of the hypotenuse upon the other side is 36 rods. Find the distance around the lot.

12. In a triangle, the projection of either leg upon the altitude drawn to the hypotenuse is 8′. The hypotenuse is 16′. Find the angles and sides of the triangle and the segments of the hypotenuse.

13. Divide a one-inch line into seven equal parts.

14. Divide a one-inch line into nine equal parts.

15. The segments of the bisectors of the base angles of an isosceles triangle form a proportion. What is the ratio of the two greater segments?

16. The segments of the altitudes of an equilateral triangle may form a proportion. Find the ratio in this proportion.

17. A lamp-post is 8 feet high and at noon casts a shadow $3\frac{1}{2}$ feet long. At the same time a tree casts a shadow 21 feet long. Find the height of the tree.

THEOREM XLVIII

195. *In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other side upon it.*

Draw $\triangle ABC$, either acute-angled or obtuse-angled at B.

Draw $CE \perp AB$. Then AE (p) is the projection of AC upon AB.

Call AC (b); CB (a); EB (x); CE (h); AB (c). We now have

Given $\triangle ABC$ with acute angle at A.

To prove that $a^2 = b^2 + c^2 - 2\,pc$.

Proof. 1. (Start this proof with a^2, in $\triangle EBC$.)

$$a^2 = h^2 + x^2.$$

2. But $x = c - p$, if $\angle B$ is acute.
$x = p - c$, if $\angle B$ is obtuse.

3. Then, $x^2 = c^2 + p^2 - 2\,pc$. (Type II, § 160.)

4. And $a^2 = h^2 + c^2 + p^2 - 2\,pc$. (From 1.)

5. But $h^2 + p^2 = b^2$.

6. Hence, $a^2 = b^2 + c^2 - 2\,pc$.

THEOREM XLIX

196. *In any obtuse-angled triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides and the projection of the other side upon it.*

Draw $\triangle ABC$, obtuse-angled at B. Let the projection of BC (a) be BE (p). Call CE (h); AE (x). We now have

Given $\triangle ABC$, obtuse-angled at B.

To prove $b^2 = a^2 + c^2 + 2\,pc$.

Proof. 1. Start, as in § 195, with the square of the side wanted.

$$b^2 = h^2 + x^2.$$

2. But $x = c + p.$

(Then proceed as in § 195.)

EXERCISE 68

1. In any triangle, if a perpendicular is drawn from the vertex to the base, the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.

2. In any triangle, if an altitude is drawn, the ratio of the sum of the sides to the sum of the segments of the base is the same as the ratio of the difference of the segments of the base to the difference of the sides.

3. In a right triangle ABC, right-angled at C, $a^2 = 3b^2$. A median is drawn from C to M. Show that $\triangle AMC$ is equilateral.

4. In any triangle, if a median is drawn to the base, the sum of the squares of the sides is equal to twice the square of the median, plus twice the square of half the base. (When the median is drawn, acute-angled and obtuse-angled triangles are formed. Project the median upon the base and use §§ 195, 196.)

5. The sides of a triangle are 14′, 18′, 24′. A median is drawn to side 24′. Find the projection of the median upon this side.

6. In any triangle, if a median is drawn to the base, the difference of the squares of the other two sides is equal to twice the product of the base and the projection of the median upon the base.

7. The sides of a triangle are 15″, 12″, 9″. A median is drawn to side 12″. Find the length of the median.

8. In example 7, find the altitude drawn to side 12″.

9. A median drawn to the base of a triangle is 26′, the sides are 20′ and 48′, respectively. Find the base and area.

10. The hypotenuse of a right triangle is 34. Find the median drawn to the hypotenuse. Will more than one triangle satisfy this condition?

11. The hypotenuse of a right triangle is $2c$. Find the median drawn to the hypotenuse.

12. One side of an isosceles right triangle is 18″. Find the median drawn to the hypotenuse.

13. In example 11, if the center of a circle were the middle point of the hypotenuse and the radius were c, where would the vertices of the triangle lie?

14. The sides of a triangle are 6″, 3″, 8″. Is the angle opposite side 8″ right, acute, or obtuse?

15. The sides of a triangle are 15, 16, 17. Is the triangle acute-angled or obtuse-angled?

16. The sides of a triangle are 14″, 15″, 20″. What kind of a triangle is it?

17. The sides of a triangle are 14″, 15″, 21″. What kind of a triangle is it? What kind of an angle is opposite side 21″? Side 15″?

18. The sides of a triangle are 14, 15, $\sqrt{421}$. What kind of a triangle is it?

19. The sides of a triangle are 18″, 24″, 30″, and an altitude is drawn to base 30″. Find the segments into which this altitude divides the base. Find this altitude.

20. The sides of a triangle are 12′–6″, 30′–0″, 32′–6″. The area is $187\frac{1}{2}$ square feet. Find the three altitudes and the three angles of the triangle.

21. Theorems XLVII, XLVIII, XLIX are closely related. If a, b, c, are the sides of a triangle, and p the projection of a side, show the relation between

$$b^2 = a^2 + c^2 - 2\,pc,$$
$$b^2 = a^2 + c^2,$$
$$b^2 = a^2 + c^2 + 2\,pc,$$

and what change in angles will produce any of the three equations from the other.

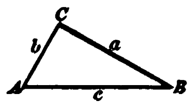
22. In a given triangle $a = 28$, $b = 35$, $c = 21$. Medians are drawn from A, B, and C. Find the distance from each vertex to the intersection of the medians (§ 146).

23. The sides of a triangle are 7, 8, 9. Find the distance from the vertex opposite side 9 to the intersection of the medians. (Example 4, Exercise 68.)

197. The ratios of sides and the hypotenuse of a right triangle play an important part in the solving of triangles, and in surveying.

For purposes of convenience these ratios have been given special names.

Draw a triangle ABC right-angled at C. Name the sides as usual. We use the following *definitions:*



Tangent A is the ratio of the side opposite the angle A to the side adjacent to the angle, or $\frac{a}{b}$.

Sine A is the ratio of the opposite side to the hypotenuse, or $\frac{a}{c}$.

Cosine A is the ratio of the adjacent side to the hypotenuse.

These expressions are abbreviated: $\begin{cases} \tan A = \dfrac{a}{b}. \\ \sin A = \dfrac{a}{c}. \\ \cos A = \dfrac{b}{c}. \end{cases}$

These are called *trigonometric ratios.*

Find the ratios for tan B, sin B, cos B, in this same triangle.

EXERCISE 69

Solve the following equations:

1. $4 = \frac{3}{x}$. 2. $7 = \frac{9}{b}$. 3. $6 = \frac{a}{7}$.

4. $1.2341 = \frac{18}{b}$. 5. $.69934 = \frac{25}{c}$.

6. Solve for c: $\sin A = \frac{20}{c}$. *Ans.* $c = \frac{20}{\sin A}$.

7. Solve for c: $\sin 30° \, 38' = \frac{41}{c}$.

8. Solve for a: $\tan 47° = \frac{a}{21}$.

9. Solve for b: $\tan 24° \, 19' = \frac{210}{b}$.

10. $a = 3$, $b = 4$. Find $\tan A$, $\sin A$, $\cos A$.

$$c = \sqrt{a^2 + b^2} \quad (\S\ 193)$$
$$= \sqrt{3^2 + 4^2}$$
$$= 5.$$
$$\tan A = \frac{a}{b} = \frac{3}{4} = .75.$$
$$\sin A = \frac{a}{c} = \frac{3}{5} = .6.$$
$$\cos A = \frac{b}{c} = \frac{4}{5} = .8.$$

11. Express decimally the ratio $\frac{15}{18}$. (Five decimal places.)

12. Express decimally the ratio $\frac{5}{80}$.

13. Express decimally the ratio $\frac{5}{6}$.

(Remember that .625 is just as much a ratio as $\frac{625}{1000}$, or as $\frac{5}{8}$. The tangent, sine, and cosine ratios are always reduced to decimal forms.)

14. $a = 8$, $c = 16$. Find sine A. How many degrees in $\angle A$? in $\angle B$?

15. $a = 42$, $b = 42$. Find tangent A. How many degrees in $\angle A$?

16. $\angle A = 30°$, $a = 16'$. Find c.

In example 14, we learned that

$$\sin 30° = .5,$$

$$\sin A = \frac{a}{c}.$$

Substituting the given values of $\angle A$ and a,

$$\sin 30° = \frac{16'}{c},$$

or,
$$.5 = \frac{16'}{c}.$$

Then,
$$.5\,c = 16'.$$

$$c = \frac{16'}{.5} = 32'.$$

17. $A = 45°$, $a = 22$. Find b.

18. In example 17, $a = b$, and $\sqrt{a^2 + b^2} = c$.

Hence,
$$c = \sqrt{22^2 + 22^2} = \sqrt{22^2(1+1)} = \sqrt{22^2 \cdot 2}$$
$$= \sqrt{22^2} \cdot \sqrt{2} = 22\sqrt{2}.$$

Or the diagonal of a square equals its side times the square root of 2. If $s =$ one side of a square, *diagonal* $= s\sqrt{2}$.

19. One angle of a right triangle is 45°. $c = 14\sqrt{2}$. Find a and b.

20. $a = 25$, $c = 42$. Find $\sin A$.

21. $b = 19$, $c = 30$. Find $\cos A$.

22. In example 20, find $\cos B$.

23. In example 21, find $\sin B$.

24. Is the sine a proper or an improper fraction? Why?

25. Is the cosine less or greater than one?

26. Is the tangent a proper or an improper fraction?

198. Tables have been compiled containing the ratios for sine, cosine, and tangent for all angles. Those on pages 200–203 are prepared for intervals of 10 minutes for each degree.

Ex. 1. Find the sine of 30°.

On the page marked natural sines look down the column headed DEG. until 30° is reached. The second column is for 0′. Hence the sine of 30° is .50000. The truth of this ratio can be easily shown geometrically. Draw right triangle ABC, with $\angle A = 30°$, $\angle B = 60°$. In this triangle, $c = 2a$. Then sine $A = \frac{a}{c} = \frac{1}{2} = .5$.

Ex. 2. Find sin 30° 40′.

Read down the degree column to 30°. Then read across to the column headed 40′. The sine of 30° 40′ is seen to be .51004, which is the ratio of $\frac{a}{c}$ when $\angle A = 30° 40'$.

Ex. 3. Find sin 30° 42′.

$$\text{sine } 30° 40' = .51004$$
$$\text{sine } 30° 50' = .51254$$

The difference between these ratios is .00250. That is, for this part of the table it takes .00250 increase to make 10′. But we have only 2′ more than 40′. Hence, we must add $\frac{2}{10}$ or $\frac{1}{5}$ of .00250 to .51004. That is, the sin 30° 42′ = sin 30° 40′ + $\frac{1}{5}$ of .00250.

The written work should stand:

$$\begin{aligned} \sin 30° 40' &= .51004 \\ \text{Correction for } 2' &= \underline{.00050} \\ \sin 30° 42' &= .51054 \end{aligned}$$

Ex. 4. Find tan 30° 40′.

On the first page marked natural tangents read down the degree column until 30° is reached, then read across to the column headed 40.

The tan 30° 40′ = .59297.

Ex. 5. Find tan 30° 42′.

$$\begin{aligned} \tan 30° 40' &= .59297 \\ \tan 30° 50' &= \underline{.59690} \\ \text{Correction for } 10' &= .00393 \\ \text{Correction for } 2' &= \tfrac{1}{5} \text{ of } .00393 \\ &= .00078 \end{aligned}$$

Hence, tan 30° 42′ = .59297 + .00078 = .59375.

Ex. 6. The sine of an angle is .86602. Find the angle.

Looking at the sine ratios, we see that they increase from 0° to 90°. On the second page of the sines we find that .86602 is the sine of 60°.

Ex. 7. The sine of an angle is .56321. Find the angle.

The next lowest sine ratio to .56321 is .56160; the next highest is .56400. Our angle evidently lies between 34° 10′ and 34° 20′.

$$\begin{aligned} \sin 34^\circ\, 20' &= .56400 \\ \sin 34^\circ\, 10' &= \underline{.56160} \\ & \ .00240 \end{aligned}$$

It takes .00240 to make 10′. We have .56321 − .56160 or .00161; that is, our angle is $34^\circ\, 10' + \frac{.00161}{.00240}$ of $10' = 34^\circ\, 10' + 6.7' = 34^\circ\, 16.7' = 34^\circ\, 16'\, 42''$.

Ex. 8. The cosine of an angle is .84596. Find the angle.

In finding cosines, read up on the sine table. Note that the *cosine ratio decreases* as the angle increases. The correction for cosines must therefore be subtracted. The next lowest ratio to .84596 is .84495.

$$\begin{aligned} \text{cosine } 32^\circ\, 20' &= .84495 \\ \text{cosine } 32^\circ\, 10' &= \underline{.84650} \\ \text{Correction for } 10' &= .00155 \\ .84596 - .84495 &= .00101 \end{aligned}$$

Hence our angle, whose cosine is .84596, is $32^\circ\, 20' - \frac{101}{155}$ of 10′, or $32^\circ\, 20' - 6.516'$ or $32^\circ\, 13'\, 29''$.

EXERCISE 70

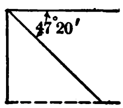
1. $a = 8$, $c = 20$. Find $\angle A$.
2. Sin $A = .70710$, $a = 24$. Find c and $\angle A$.
3. Sin $B = .5$, $c = 38''$. Find b, $\angle B$, $\angle A$.
4. Sin $A = .70710$, $a = 43'$. Find c, $\angle B$, $\angle A$.
5. Sin $A = .88294$, $c = 75'$. Find a, $\angle A$, $\angle B$.
6. In $\triangle ABC$, $a = 16$, $b = 22$. Find $\angle A$, $\angle B$, c.
7. $b = 27$, $c = 41$. Find $\angle A$, $\angle B$, a.
8. Tan $A = 2.3558$, $b = 73$. Find a, $\angle B$, $\angle A$, c.
9. Tan $B = .48773$, $a = 73'$. Find $\angle$ B, b, $\angle$ A, c.

10. Tan $B = .43827$, $a = 275'$. Find b, $\angle B$, $\angle A$, c.

11. $\angle B = 38° \ 30'$, $c = 371'$. Find $\angle A$, a, b.

12. At a point 75 feet from the foot of a flag pole, the angle of elevation of the top of the pole is 32°. Find the height of the pole.*

13. From the top of an electric light tower 125′ high, the angle of depression of a man standing on the ground is 47° 20′. Find the distance of the man from the foot of the tower.



14. A telephone pole 33′ high casts a shadow 40′ long. Find the angle of elevation of the sun.

15. The width of a house is $26\frac{1}{2}$ feet. The height of the eaves above the ground is 31 feet, 4 inches, and the height of the ridge pole above the ground is 41 feet. Find the slope of the roof and the approximate length of the rafters.

16. One angle at the base of an isosceles triangle is 28°, the base is 32′. Find the altitude and legs of the triangle.

17. At two o'clock on March 15, the angle of elevation of the sun is 37° 8.2′. The distance from the point directly under a tower to the shadow of the top of the tower is 117′. Find the height of the tower.

18. Two parallels AB and CD are cut by a transversal at E and F. EF is 20″ and makes an angle of 120° with AB. The interior angles at E and F are bisected. Find the area of the quadrilateral formed by these bisectors.

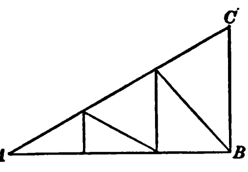
19. In example 18, find the area when EF makes an angle of 100° with AB.

20. The perimeter of a parallelogram is 50″, one side is 8″ and one angle 38°. Find the area of the parallelogram.

21. The distance between parallel sides of a square steel nut is $\frac{5}{8}$″. Find the diameter of the cylinder from which it was cut.

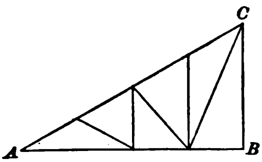
* Angle of elevation is from the horizontal up, angle of depression is from *horizontal down.*

22. At a point B the angle of elevation of the top of a tower is 60°; at a point A, 100 feet farther from the base of the tower the angle of elevation is 30°. Find the height of the tower. (From figure get expressions for tan 30° and tan 60°. Solve these as simultaneous equations.)



23. In a roof truss ABC, the slope is 30°, and $BC = 8'$. Find the length of angle irons which brace the roof. (AB is trisected.)

24. In a roof truss ABC, AB is in four equal parts. The pitch of the roof is 30° and each section of AC is $5'$. Find the lengths of the braces and of BC.



25. In a roof truss of the form of example 23, the distance from eave to eave is $52'$. The pitch of the roof is $\frac{1}{4}$. Find the lengths of all angle irons used.

26. At a point B, the angle of elevation of the top of a tree is 61° 30′. At a point $200'$ farther from the tree, the angle of elevation is 28° 20′. Find the height of the tree.

199. Fill out the following synopsis:

Two triangles are similar when:

1.

2.

3.

4.

5.

Two right triangles are similar when:

NATURAL SINES — (Read down)

DEG.	0′	10′	20′	30′	40′	50′	60′	
0	.00000	.00291	.00581	.00872	.01163	.01454	.01745	89
1	.01745	.02036	.02326	.02617	.02908	.03199	.03489	88
2	.03489	.03780	.04071	.04361	.04652	.04943	.05233	87
3	.05233	.05524	.05814	.06104	.06395	.06685	.06975	86
4	.06975	.07265	.07555	.07845	.08135	.08425	.08715	85
5	.08715	.09005	.09295	.09584	.09874	.10163	.10452	84
6	.10452	.10742	.11031	.11320	.11609	.11898	.12186	83
7	.12186	.12475	.12764	.13052	.13341	.13629	.13917	82
8	.13917	.14205	.14493	.14780	.15068	.15356	.15643	81
9	.15643	.15930	.16217	.16504	.16791	.17078	.17364	80
10	.17364	.17651	.17937	.18223	.18509	.18795	.19080	79
11	.19080	.19366	.19651	.19936	.20221	.20506	.20791	78
12	.20791	.21075	.21359	.21644	.21927	.22211	.22495	77
13	.22495	.22778	.23061	.23344	.23627	.23909	.24192	76
14	.24192	.24474	.24756	.25038	.25319	.25600	.25881	75
15	.25881	.26162	.26443	.26723	.27004	.27284	.27563	74
16	.27563	.27843	.28122	.28401	.28680	.28958	.29237	73
17	.29237	.29515	.29793	.30070	.30347	.30624	.30901	72
18	.30901	.31178	.31454	.31730	.32006	.32281	.32556	71
19	.32556	.32831	.33106	.33380	.33654	.33928	.34202	70
20	.34202	.34475	.34748	.35020	.35293	.35565	.35836	69
21	.35836	.36108	.36379	.36650	.36920	.37190	.37460	68
22	.37460	.37730	.37999	.38268	.38536	.38805	.39073	67
23	.39073	.39340	.39607	.39874	.40141	.40407	.40673	66
24	.40673	.40939	.41204	.41469	.41733	.41998	.42261	65
25	.42261	.42525	.42788	.43051	.43313	.43575	.43837	64
26	.43837	.44098	.44359	.44619	.44879	.45139	.45399	63
27	.45399	.45658	.45916	.46174	.46432	.46690	.46947	62
28	.46947	.47203	.47460	.47715	.47971	.48226	.48481	61
29	.48481	.48735	.48989	.49242	.49495	.49747	.50000	60
30	.50000	.50251	.50503	.50753	.51004	.51254	.51503	59
31	.51503	.51752	.52001	.52249	.52497	.52745	.52991	58
32	.52991	.53238	.53484	.53730	.53975	.54219	.54463	57
33	.54463	.54707	.54950	.55193	.55436	.55677	.55919	56
34	.55919	.56160	.56400	.56640	.56880	.57119	.57357	55
35	.57357	.57595	.57833	.58070	.58306	.58542	.58778	54
36	.58778	.59013	.59248	.59482	.59715	.59948	.60181	53
37	.60181	.60413	.60645	.60876	.61106	.61336	.61566	52
38	.61566	.61795	.62023	.62251	.62478	.62705	.62932	51
39	.62932	.63157	.63383	.63607	.63832	.64055	.64278	50
40	.64278	.64501	.64723	.64944	.65165	.65386	.65605	49
41	.65605	.65825	.66043	.66262	.66479	.66696	.66913	48
42	.66913	.67128	.67344	.67559	.67773	.67986	.68199	47
43	.68199	.68412	.68624	.68835	.69046	.69256	.69465	46
44	.69465	.69674	.69883	.70090	.70298	.70504	.70710	45
	60′	50′	40′	30′	20′	10′	0′	DEG.

NATURAL COSINES — (Read up)

Natural Sines — (Read down)

Deg.	0′	10′	20′	30′	40′	50′	60′	
45	.70710	.70916	.71120	.71325	.71528	.71731	.71934	44
46	.71934	.72135	.72336	.72537	.72737	.72936	.73135	43
47	.73135	.73333	.73530	.73727	.73923	.74119	.74314	42
48	.74314	.74508	.74702	.74895	.75088	.75279	.75471	41
49	.75471	.75661	.75851	.76040	.76229	.76417	.76604	40
50	.76604	.76791	.76977	.77162	.77347	.77531	.77714	39
51	.77714	.77897	.78079	.78260	.78441	.78621	.78801	38
52	.78801	.78979	.79157	.79335	.79512	.79688	.79863	37
53	.79863	.80038	.80212	.80385	.80558	.80730	.80901	36
54	.80901	.81072	.81242	.81411	.81580	.81748	.81915	35
55	.81915	.82081	.82247	.82412	.82577	.82740	.82903	34
56	.82903	.83066	.83227	.83388	.83548	.83708	.83867	33
57	.83867	.84025	.84182	.84339	.84495	.84650	.84804	32
58	.84804	.84958	.85111	.85264	.85415	.85566	.85716	31
59	.85716	.85866	.86014	.86162	.86310	.86456	.86602	30
60	.86602	.86747	.86892	.87035	.87178	.87320	.87462	29
61	.87462	.87602	.87742	.87881	.88020	.88157	.88294	28
62	.88294	.88430	.88566	.88701	.88835	.88968	.89100	27
63	.89100	.89232	.89363	.89493	.89622	.89751	.89879	26
64	.89879	.90006	.90132	.90258	.90383	.90507	.90630	25
65	.90630	.90753	.90875	.90996	.91116	.91235	.91354	24
66	.91354	.91472	.91589	.91706	.91821	.91936	.92050	23
67	.92050	.92163	.92276	.92388	.92498	.92609	.92718	22
68	.92718	.92827	.92934	.93041	.93148	.93253	.93358	21
69	.93358	.93461	.93565	.93667	.93768	.93869	.93969	20
70	.93969	.94068	.94166	.94264	.94360	.94456	.94551	19
71	.94551	.94646	.94739	.94832	.94924	.95015	.95105	18
72	.95105	.95195	.95283	.95371	.95458	.95545	.95630	17
73	.95630	.95715	.95799	.95882	.95964	.96045	.96126	16
74	.96126	.96205	.96284	.96363	.96440	.96516	.96592	15
75	.96592	.96667	.96741	.96814	.96887	.96958	.97029	14
76	.97029	.97099	.97168	.97237	.97304	.97371	.97437	13
77	.97437	.97502	.97566	.97629	.97692	.97753	.97814	12
78	.97814	.97874	.97934	.97992	.98050	.98106	.98162	11
79	.98162	.98217	.98272	.98325	.98378	.98429	.98480	10
80	.98480	.98530	.98580	.98628	.98676	.98722	.98768	9
81	.98768	.98813	.98858	.98901	.98944	.98985	.99026	8
82	.99026	.99066	.99106	.99144	.99182	.99218	.99254	7
83	.99254	.99289	.99323	.99357	.99389	.99421	.99452	6
84	.99452	.99482	.99511	.99539	.99567	.99593	.99619	5
85	.99619	.99644	.99668	.99691	.99714	.99735	.99756	4
86	.99756	.99776	.99795	.99813	.99830	.99847	.99863	3
87	.99863	.99877	.99891	.99904	.99917	.99928	.99939	2
88	.99939	.99948	.99957	.99965	.99972	.99979	.99984	1
89	.99984	.99989	.99993	.99996	.99998	.99999	1.0000	0
	60′	50′	40′	30′	20′	10′	0′	Deg.

Natural Cosines — (Read up)

Natural Tangents — (Read down)

Deg.	0′	10′	20′	30′	40′	50′	60′	
0	.00000	.00290	.00581	.00872	.01163	.01454	.01745	89
1	.01745	.02036	.02327	.02618	.02909	.03200	.03492	88
2	.03492	.03783	.04074	.04366	.04657	.04949	.05240	87
3	.05240	.05532	.05824	.06116	.06408	.06700	.06992	86
4	.06992	.07285	.07577	.07870	.08162	.08455	.08748	85
5	.08748	.09042	.09335	.09628	.09922	.10216	.10510	84
6	.10510	.10804	.11099	.11393	.11688	.11983	.12278	83
7	.12278	.12573	.12869	.13165	.13461	.13757	.14054	82
8	.14054	.14350	.14647	.14945	.15242	.15540	.15838	81
9	.15838	.16136	.16435	.16734	.17033	.17332	.17632	80
10	.17632	.17932	.18233	.18533	.18834	.19136	.19438	79
11	.19438	.19740	.20042	.20345	.20648	.20951	.21255	78
12	.21255	.21559	.21864	.22169	.22474	.22780	.23086	77
13	.23086	.23393	.23700	.24007	.24315	.24624	.24932	76
14	.24932	.25242	.25551	.25861	.26172	.26483	.26794	75
15	.26794	.27106	.27419	.27732	.28046	.28360	.28674	74
16	.28674	.28989	.29305	.29621	.29938	.30255	.30573	73
17	.30573	.30891	.31210	.31529	.31850	.32170	.32492	72
18	.32492	.32813	.33136	.33459	.33783	.34107	.34432	71
19	.34432	.34758	.35084	.35411	.35739	.36067	.36397	70
20	.36397	.36726	.37057	.37388	.37720	.38053	.38386	69
21	.38386	.38720	.39055	.39391	.39727	.40064	.40402	68
22	.40402	.40741	.41080	.41421	.41762	.42104	.42447	67
23	.42447	.42791	.43135	.43481	.43827	.44174	.44522	66
24	.44522	.44871	.45221	.45572	.45924	.46277	.46630	65
25	.46630	.46985	.47341	.47697	.48055	.48413	.48773	64
26	.48773	.49133	.49495	.49858	.50221	.50586	.50952	63
27	.50952	.51319	.51687	.52056	.52427	.52798	.53170	62
28	.53170	.53544	.53919	.54295	.54672	.55051	.55430	61
29	.55430	.55811	.56193	.56577	.56961	.57347	.57735	60
30	.57735	.58123	.58513	.58904	.59297	.59690	.60086	59
31	.60086	.60482	.60880	.61280	.61680	.62083	.62486	58
32	.62486	.62892	.63298	.63707	.64116	.64528	.64940	57
33	.64940	.65355	.65771	.66188	.66607	.67028	.67450	56
34	.67450	.67874	.68300	.68728	.69157	.69588	.70020	55
35	.70020	.70455	.70891	.71329	.71769	.72210	.72654	54
36	.72654	.73099	.73546	.73996	.74447	.74900	.75355	53
37	.75355	.75812	.76271	.76732	.77195	.77661	.78128	52
38	.78128	.78598	.79069	.79543	.80019	.80497	.80978	51
39	.80978	.81461	.81946	.82433	.82923	.83415	.83910	50
40	.83910	.84406	.84906	.85408	.85912	.86419	.86928	49
41	.86928	.87440	.87955	.88472	.88992	.89515	.90040	48
42	.90040	.90568	.91099	.91633	.92169	.92709	.93251	47
43	.93251	.93796	.94345	.94896	.95450	.96008	.96568	46
44	.96568	.97132	.97699	.98269	.98843	.99419	1.0000	45
	60′	50′	40′	30′	20′	10′	0′	Deg.

Natural Cotangents — (Read up)

NATURAL TANGENTS — (Read down)

DEG.	0′	10′	20′	30′	40′	50′	60′	
45	1.0000	1.0058	1.0117	1.0176	1.0235	1.0295	1.0355	44
46	1.0355	1.0415	1.0476	1.0537	1.0599	1.0661	1.0723	43
47	1.0723	1.0786	1.0849	1.0913	1.0977	1.1041	1.1106	42
48	1.1106	1.1171	1.1236	1.1302	1.1369	1.1436	1.1503	41
49	1.1503	1.1571	1.1639	1.1708	1.1777	1.1847	1.1917	40
50	1.1917	1.1988	1.2059	1.2131	1.2203	1.2275	1.2349	39
51	1.2349	1.2422	1.2496	1.2571	1.2647	1.2723	1.2799	38
52	1.2799	1.2876	1.2954	1.3032	1.3111	1.3190	1.3270	37
53	1.3270	1.3351	1.3432	1.3514	1.3596	1.3680	1.3763	36
54	1.3763	1.3848	1.3933	1.4019	1.4106	1.4193	1.4281	35
55	1.4281	1.4370	1.4459	1.4550	1.4641	1.4733	1.4825	34
56	1.4825	1.4919	1.5013	1.5108	1.5204	1.5301	1.5398	33
57	1.5398	1.5497	1.5596	1.5696	1.5798	1.5900	1.6003	32
58	1.6003	1.6107	1.6212	1.6318	1.6425	1.6533	1.6642	31
59	1.6642	1.6753	1.6864	1.6976	1.7090	1.7204	1.7320	30
60	1.7320	1.7437	1.7555	1.7674	1.7795	1.7917	1.8040	29
61	1.8040	1.8164	1.8290	1.8417	1.8546	1.8676	1.8807	28
62	1.8807	1.8940	1.9074	1.9209	1.9347	1.9485	1.9626	27
63	1.9626	1.9768	1.9911	2.0056	2.0203	2.0352	2.0503	26
64	2.0503	2.0655	2.0809	2.0965	2.1123	2.1283	2.1445	25
65	2.1445	2.1609	2.1774	2.1943	2.2113	2.2285	2.2460	24
66	2.2460	2.2637	2.2816	2.2998	2.3182	2.3369	2.3558	23
67	2.3558	2.3750	2.3944	2.4142	2.4342	2.4545	2.4750	22
68	2.4750	2.4959	2.5171	2.5386	2.5604	2.5826	2.6050	21
69	2.6050	2.6279	2.6510	2.6746	2.6985	2.7228	2.7474	20
70	2.7474	2.7725	2.7980	2.8239	2.8502	2.8770	2.9042	19
71	2.9042	2.9318	2.9600	2.9886	3.0178	3.0474	3.0776	18
72	3.0776	3.1084	3.1397	3.1715	3.2040	3.2371	3.2708	17
73	3.2708	3.3052	3.3402	3.3759	3.4123	3.4495	3.4874	16
74	3.4874	3.5260	3.5655	3.6058	3.6470	3.6890	3.7320	15
75	3.7320	3.7759	3.8208	3.8667	3.9136	3.9616	4.0107	14
76	4.0107	4.0610	4.1125	4.1653	4.2193	4.2747	4.3314	13
77	4.3314	4.3896	4.4494	4.5107	4.5736	4.6382	4.7046	12
78	4.7046	4.7728	4.8430	4.9151	4.9894	5.0658	5.1445	11
79	5.1445	5.2256	5.3092	5.3955	5.4845	5.5763	5.6712	10
80	5 6712	5.7693	5.8708	5.9757	6.0844	6.1970	6.3137	9
81	6.3137	6.4348	6.5605	6.6911	6.8269	6.9682	7.1153	8
82	7.1153	7.2687	7.4287	7.5957	7.7703	7.9530	8.1443	7
83	8.1443	8.3449	8.5555	8.7768	9.0098	9.2553	9.5143	6
84	9.5143	9.7881	10.078	10.385	10.711	11.059	11.430	5
85	11.430	11.826	12.250	12.706	13.196	13.726	14.300	4
86	14.300	14.924	15.604	16.349	17.169	18.075	19.081	3
87	19.081	20.205	21.470	22.904	24.541	26.431	28.636	2
88	28.636	31.241	34.367	38.188	42.964	49.103	57.290	1
89	57.290	68.750	85.939	114.58	171.88	343.77	∞	0
	60′	50′	40′	30′	20′	10′	0′	DEG.

NATURAL COTANGENTS — (Read up)

CHAPTER XI

Measurement

200. *Measurement* depends upon ratio. It is usually the ratio of one quantity to another quantity taken as a unit.

Up to this point relations of lines have been studied. Theorems I–XXXIV involved position and equality of lines. Theorems XXXV–XLIX deal with measurement (ratios of lengths) of lines. In the problems relating to chemistry and physics measurement (ratios) of weights and volumes appeared. We now deal with another ratio, that of surfaces.

A unit of surface is usually a square, one unit (1′, 1″, etc.) on a side.

Theorem L

201. *Two rectangles with equal altitudes have the same ratio as their bases.*

When we speak of a rectangle in this sense we mean its area, its surface, the number of times its surface contains the *surface unit* of measure. That is, the ratio of surface to the unit of surface.

Draw rectangles $ABCD$ and $EFGH$, having altitude AD equal to altitude EH. We have

Given rectangles $ABCD$ and $EFGH$ having a common altitude AD.

To prove $\dfrac{ABCD}{EFGH}=\dfrac{AB}{EF}$.

Proof. 1. Take some unit of measure AM and let it be contained in AB say 4 times, and in EF, 3 times. On AB and EF at the extremities of the units, AM, applied to these bases erect ⊥s to AB and EF.

2. $ABCD$ is now divided into four equal rectangles and $EFGH$ into three equal rectangles.

3. Then, $\dfrac{ABCD}{EFGH}=\dfrac{4}{3}.$

4. But $\dfrac{AB}{EF}=\dfrac{4}{3}.$

5. Hence, $\dfrac{ABCD}{EFGH}=\dfrac{AB}{EF}.$

EXERCISE 71

Prove that two rectangles having equal bases are in the same ratio as their altitudes.

THEOREM LI

202. *Any two rectangles have the same ratio as the products of their bases by their altitudes.*

Draw rectangle M with base and altitude b and a, respectively. Draw rectangle N with base and altitude b' and a', respectively. We have

Given rectangles M and N, whose dimensions are b, a, and b', a', respectively.

To prove $\dfrac{M}{N}=\dfrac{ab}{a'b'}.$

Proof. 1. Draw rectangle K with base b' and altitude a.

2. Then, $\dfrac{M}{K}=\dfrac{b}{b'}.$ (§ 201.)

3. And $\dfrac{K}{N}=\dfrac{a}{a'}.$ (Exercise 71.)

4. Multiplying 2 by 3,

$$\frac{M}{N}=\frac{ab}{a'b'}.$$

203. It follows from § 202 that if N is the unit of surface $a'=1$ and $b'=1$, step 4 of § 202 becomes

$$\frac{M}{N}\simeq\frac{ab}{1}.$$

But N is the unit of surface,

then, $\frac{M}{N} = M$, and $M = ab$.

That is, *the area of a rectangle is equal to the product of the base by the altitude.*

204. If two surfaces have the same area, they are *equivalent* ($\Leftrightarrow$).

THEOREM LII

205. *The area of a parallelogram is equal to the product of its base and altitude.*

Draw $\square$ $ABCD$. Draw $DE \perp AB$. Call AB, b; and DE, a. We have

Given $\square$ $ABCD$ with base b and altitude a.

To prove area $ABCD = a \cdot b$.

Proof. 1. Draw $CF \perp AB$ produced at F.

2. $\triangle AED = \triangle BFC$. (?)

3. Then rectangle $EFCD$ and $\square$ $ABCD$ are equivalent and have the same base and altitude.

4. But area $EFCD = a \cdot b$.

5. Hence area $ABCD = a \cdot b$.

206. We may decide from § 205 that:

1. Two parallelograms having equal bases and altitudes are equivalent.

2. Two parallelograms having equal altitudes have the same ratio as their bases.

3. Two parallelograms having equal bases have the same ratio as their altitudes.

4. Two parallelograms are in the same ratio as the products of their bases and altitudes.

Note that in all theorems involving lines no products of *lines were involved* in the result. In all *area theorems* the

products of two lines are involved in every ratio. This is because a line has one dimension and a surface has two dimensions. If we studied volumes, we would find that in *volume theorems* the ratios involved the products of three lines.

THEOREM LIII

207. *The area of a triangle is equal to one half the product of the base and altitude.*

Draw $\triangle ABC$. Through C draw $\parallel$ to AB. Through B draw $\parallel$ to AC. A parallelogram is now formed whose diagonal is CB. (The proof is left to the pupil.)

208. From § 207 we conclude that:

1. Two triangles having equal bases and altitudes are equivalent. (Are they necessarily equal?)
2. Two triangles with equal altitudes are in the same ratio as their bases.
3. Two triangles with equal bases are in the same ratio as their altitudes.
4. Two triangles have the same ratio as the products of their bases and altitudes.

EXERCISE 72

1. The side of a square is 5. Find the diagonal.

2. The side of a square is s. Find the diagonal.

3. The side of an equilateral triangle is 16. Find the altitude.

4. The side of an equilateral triangle is 10. Find the altitude. Find the area. Find the altitude when the side is 5.

5. The diagonal of a square is $12\sqrt{2}$. Find the side.

6. The diagonal of a square is 12. Find the side.

7. The diagonal of a square is d. Find the side.

8. The side of an equilateral triangle is a. Find the altitude.

The altitude divides the triangle into two equal right triangles. The base and hypotenuse of one of these triangles are $\frac{a}{2}$ and a, respectively. The altitude h is found from the equation,

$$h^2 = a^2 - \left(\frac{a}{2}\right)^2$$

$$= \frac{3\,a^2}{4}$$

$$= \frac{a^2}{4} \cdot 3.$$

$$h = \sqrt{\frac{a^2}{4} \cdot 3}$$

$$= \sqrt{\frac{a^2}{4}} \cdot \sqrt{3}$$

$$= \frac{a}{2}\sqrt{3}.$$

The results of examples 1, 2, 3, 4, 5, 6, 7, 8, should be put into forms involving $\sqrt{2}$ or $\sqrt{3}$.

We may then state:

(a) The diagonal of a square is equal to the side times $\sqrt{2}$.

(b) The altitude of an equilateral triangle is equal to one half the side times $\sqrt{3}$.

9. The base of an equilateral triangle is 8, the altitude is $4\sqrt{3}$. Find the area.

10. The base of an equilateral triangle is 18. Find the area.

11. The side of an equilateral triangle is a. Find the area

12. Find $\sqrt{2}$, correct to three decimal places.

13. Find $\sqrt{3}$, correct to three decimal places.

As $\sqrt{2}$ and $\sqrt{3}$ enter all operations relating to diagonals of squares and altitudes of equilateral triangles, their decimal *values should be com*mitted to memory.

14. The side of a square is $2x+5$. Find an expression for the diagonal.

15. The area of a square is $9x^2-24x+16$. Find an expression for the diagonal. Find the diagonal correct to three decimal places when $x=2$. (Solve mentally.)

16. The area of a square is $16x^2+40x+25$. Find the diagonal when $x=-1$.

17. The side of an equilateral triangle is $4x+8$. Express the altitude. Express the area. Did you use example 11 as your formula for area?

18. The altitude of an equilateral triangle is $12\sqrt{3}$. Find the base and area.

19. The altitude of an equilateral triangle is $(x+5)\sqrt{3}$. Find the base and area.

20. The altitude of an equilateral triangle is $(2x-1)(1.732)$. Find the area when $x=2$.

21. The sides of a rectangle are 5 and 12. Find the diagonal.

22. Find five rectangles whose sides have such ratio that the expression for the value of the diagonal contains no radical.

23. The sides of a rectangle are $2x+3$ and $3x-1$. Find the diagonal.

24. The sides of a parallelogram are 8 and 20. The angle formed by these two sides is 30°. Find the area of the parallelogram.

25. The sides of a parallelogram are 14 and 21. The projection of the shorter side upon the longer is 7. Find the area of the parallelogram, correct to three decimal places.

26. The sides of a parallelogram are $2x-8$ and $14x+10$. The angle formed by these sides is 30°. Find the area of the parallelogram.

27. The sides of a parallelogram are a and b, the altitude upon b is h. $b = 5x - 9$ and the projection of a upon h is $7x - 5$. Find the area when $x = 3$.

28. In a parallelogram $ABCD$, $AD = 20$, $AB = 38$, and the projection of BC on AB produced is BE. If $AE = 38 + 10\sqrt{3}$, find the area of the parallelogram.

29. In example 28, find the area if $AD = 16$, $AB = 24$, $AE = 8(3 + \sqrt{3})$.

30. Find the ratio of the area of a square to the product of the diagonals.

31. Upon a given base and with a constant altitude how many parallelograms can be constructed?

32. With two given sides how many parallelograms can you construct?

33. How many conditions are necessary to construct one definite parallelogram?

34. Can a square and a triangle be equal?

35. Distinguish between equal and equivalent figures.

36. The area of a parallelogram is 156. The base is 13. Can you construct more than one parallelogram which will satisfy these conditions?

37. The area of a triangle is 48, the base is 12. Find the altitude.

38. If the area of a triangle is divided by the base, what is the quotient?

39. If the area of a triangle is divided by the altitude, what is the quotient?

40. In a triangle, area $\triangle = \frac{1}{2}ab$. If b is constant, what happens to area $\triangle$ if a varies? If the triangle is isosceles, draw a diagram showing your conclusions.

41. In example 40 if a is constant and b varies, what happens? If the triangle is isosceles, draw diagram showing your conclusions. Draw diagram showing results if the triangle is right-angled.

42. Would the conclusions reached in examples 40 and 41 concerning the relations between area and dimensions hold if you were dealing with rectangles or parallelograms?

43. The sides of a parallelogram are 8′ and 20′, respectively. The angle between these sides is 150°. Find area of parallelogram.

Theorem LIV

209. *The area of a trapezoid is equal to the altitude times one half the sum of the bases.*

Draw trapezoid $ABCD$, AB and DC being parallel sides. The altitude of the trapezoid is the distance (h) between the parallel sides. Let $AB = b$ and $DC = b'$. We have

Given trapezoid $ABCD$ with bases b and b' and altitude h.

To prove area $ABCD = \frac{h(b+b')}{2}$.

Proof. 1. Draw AC.

2. $\triangle ACD = \frac{hb'}{2}$. (§ 207.)

3. $\triangle ABC = \frac{hb}{2}$.

4. Then $ABCD = \triangle ACD + \triangle ABC = \frac{hb'}{2} + \frac{hb}{2}$.

5. By type IV, § 156, $\frac{hb'}{2} + \frac{hb}{2} = \frac{h}{2}(b' + b)$ or $h\left(\frac{b+b'}{2}\right)$.

EXERCISE 73

1. The bases of a trapezoid are 18 and 24, respectively. The altitude is 12. Find the area. (In area problems, always draw the figure.)

2. The bases of a trapezoid are $2x + 3$ and $4x - 9$, respectively. The altitude is $x - 9$. Find the area. Find the area and dimensions when $x = 1, 2, 9, -\frac{3}{2}$.

3. In an isosceles trapezoid the upper base is 26, one non-parallel side is 8, and its projection on the lower base is 4. Find the area of the trapezoid, correct to three decimal places.

4. In an isosceles trapezoid the upper base is 16″, one non-parallel side is 5″, and its projection on the lower base is 4″. Find the area.

5. One angle of a trapezoid is 30°. The opposite angle is 90°. The upper base is 8″. The altitude is $8\sqrt{3}$″. Find the area.

6. A median of a triangle divides the triangle into two equivalent triangles.

7. Join the middle point of a diagonal of any quadrilateral to the opposite vertices, and prove that two pairs of equivalent triangles are formed.

8. The area of a trapezoid is equal to the product of its altitude and the line joining the middle point of the non-parallel sides of the trapezoid.

9. The base angles of a trapezoid are each 60°. One non-parallel side is 18″, and the upper base is 21″. Find the area.

10. Two sides of a parallelogram are 216′ and 118′, respectively. The angle between these sides is 36°. Find the area of the parallelogram.

Let $ABCD$ be the parallelogram; $AB = 216'$ and $AD = 118'$; angle $A = 36°$, and DE (h) be the $\perp$ from D to AB.

In $\triangle AED$, $\sin A = \frac{h}{AD}$, or $\sin 36° = \frac{h}{118}$.

Then $h = 118 \cdot \sin 36°$.

When h is known the area can be found.

11. The vertical angle of an isosceles triangle is 108°. The base is 17′. Find the area.

12. In trapezoid $ABCD$, $\angle A = 54° 20'$, $\angle B = 90°$, $AB = 24'$, $AD = 14'$. Find the area of the trapezoid.

13. In an isosceles trapezoid $ABCD$, $\angle A = 64° 10'$, $AD = 15'$, $CD = 27'$. Find the dimensions and area of the trapezoid.

14. The area of an isosceles trapezoid is 162 sq. ft. The difference between the lengths of the parallel sides is 12′. The altitude is 6′. Find the dimensions and angles of the trapezoid.

15. The bases of an isosceles trapezoid are 18′ and 24′, respectively. The altitude is 12′. The middle points of four sides of the trapezoid are joined in order. Find the area of the quadrilateral formed by the lines joining these four points. (Exercise 42, 6.)

16. The vertices A, B, C, of a triangle are at the following points: $x = 2$, $y = 1$; $x = 8$, $y = 1$; $x = 5$, $y = 7$. Find the area and the angles of the triangle.

In naming points by means of their coördinates, we shall abridge the equations $x = 2$, $y = 1$, to this form (2, 1), the x coördinate always occurring first.

17. The vertices of a triangle are at $(-2, 2)$, $(8, 2)$, $(8, 8)$. Find the angles and area.

18. The vertices of a quadrilateral are at $(2, 6)$, $(10, 6)$, $(-1, -1)$, $(7, -1)$. Find the angles and the area of the quadrilateral.

19. The vertices of a quadrilateral are at $(-1, -1)$, $(13, -1)$, $(10, 6)$, $(2, 6)$. Find the angles and the area.

20. One side of a rhombus is 13. One diagonal is 24. Find the area and the angles, also the other diagonal.

21. The vertices of a triangle are at $(1, -5)$, $(1, 6)$, $(-5, 6)$. Find the area, the angles, the segments of the longest side, formed by the bisector of the greatest angle.

THEOREM LV

210. *The areas of two similar triangles have the same ratio as the squares of their homologous sides.*

Draw similar $\triangle$ ABC and DEF, side c corresponding to side f. We have

Given $\triangle ABC$ and $\triangle DEF$ similar, with f homologous to c.

To prove $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{c^2}{f^2}$.

Proof. 1. Draw altitude h to side c, and altitude h' to side f.

2. Then $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{hc}{h'f}$. (§ 208, 4.)

3. And $\dfrac{h}{h'} = \dfrac{c}{f}$. (§ 188.)

4. Then $\dfrac{c}{f}$ may be substituted for its equal $\dfrac{h}{h'}$ in 2.

And $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{c^2}{f^2}$.

Similarly it may be proved that two similar triangles have the same ratio as the squares of any two homologous lines.

THEOREM LVI

211. *Two similar polygons have the same ratio as the squares of any two homologous sides.*

Draw two polygons, one similar to the other.

Begin at a corresponding vertex in each polygon and draw all the diagonals from this point. The polygons are now divided into the same number of triangles. These triangles may be proved similar. Then use § 210. The proof is left to the pupil.

212. From theorems on similar figures we may conclude that, — in similar figures, lines are in the same ratio as homologous lines; areas are in the same ratio as squares of homologous *lines.*

THEOREM LVII

213. *Two triangles having an angle of one equal to an angle of the other have the same ratio as the products of the sides including the equal angles.*

Draw $\triangle$ ABC and DEF having $\angle A = \angle D$. We have

Given $\triangle$ ABC and DEF, with $\angle A = \angle D$.

To prove $\dfrac{\triangle ABC}{\triangle DEF} = \dfrac{AB \cdot AC}{DE \cdot DF}$.

Proof. 1. On AB lay off $AE' = DE$, and on AC lay off $AF' = DF$, and draw $E'F'$. Draw $F'B$.

2. $$\frac{\triangle AE'F'}{\triangle ABF'} = \frac{AE'}{AB}. \quad (\S\, 208, 2.)$$

3. Similarly, $$\frac{\triangle ABF'}{\triangle ABC} = \frac{AF'}{AC}.$$

4. Finding the product of 2 and 3,

$$\frac{\triangle AE'F'}{\triangle ABC} = \frac{AE' \cdot AF'}{AB \cdot AC}.$$

5. Hence, $$\frac{\triangle DEF}{\triangle ABC} = \frac{DE \cdot DF}{AB \cdot AC}. \quad (?)$$

6. Or, $$\frac{\triangle ABC}{\triangle DEF} = \frac{AB \cdot AC}{DE \cdot DF}. \quad (?)$$

EXERCISE 74

1. The areas of two similar hexagons are 14 and 98, respectively. One side of the smaller is 3. Find the homologous side of the other.

2. A side of one polygon is $3\frac{1}{2}$ and its area is 81. The corresponding side of a second similar polygon is 7. Find the area of the second polygon.

3. The sides of two squares are 3 and 9, respectively. Find the ratio of their areas.

4. The sides of two rectangles are 8, 7 and 4, 14. Find the ratio of their areas.

5. One polygon is $\frac{18}{14}$ of another similar polygon. Find the ratio of their homologous sides.

6. The surfaces of two cubes have the same ratio as the squares of their edges. The surface of the first is $\frac{27}{8}$. The surface of the second is $\frac{243}{16}$ and its edge is $\frac{9}{8}$. Find the edge of the first. Do these results check?

7. The base angles of an equilateral triangle are bisected. Show that an isosceles triangle is formed that is equivalent to one third of the equilateral triangle.

8. One side of a polygon is $10'$. An exterior angle at the extremity of this side is $60°$. The polygon is equilateral and equiangular. Find its area.

9. One angle of a rhombus is $60°$. The longest diagonal is $16\sqrt{3}$. Find the sides and area of the rhombus.

10. One side of a regular pentagon is $10'$. Find the area.

11. The longest diagonal of a regular hexagon is $14''$. Find the area of the hexagon.

12. One side of a rhombus is equal to one of its diagonals. Find an expression for the area of the rhombus.

CHAPTER XII

Involution and Evolution

214. In Chapter I we found the squares and cubes of monomial numbers and in Chapter VIII the squares of binomials. We will now extend the work to cover higher powers of both monomials and polynomials. The exponent indicates the power to which a number is to be raised and primarily shows how many times the number called **the base** is to be considered as a factor, *e.g.* $(2)^3$. Here 2 is the base and is to be used three times as a factor.

The reverse process to involution is **evolution**. Evolution is often indicated by the radical sign, $\sqrt{\ }$. This sign is an *r*, the first letter of the word *radix*, meaning root. The kind of root required is indicated by an **index** written above the radical sign. If no index appears, the second root or square root is understood.

Thus, $\sqrt{64}$ means square root of 64.

$\sqrt[3]{64}$ means cube root of 64.

$\sqrt[4]{64}$ means fourth root of 64, and so on.

The index of an **even** root is an **even** number.

The index of an **odd** root is an **odd** number.

215. The power of a fraction is obtained by raising both numerator and denominator to the required power.

216. In § 10 we learned that in squaring a number each factor of the number appeared twice as many times in the square as in the given number. It occurs three times as many

times in cubes, and so on. This is helpful in finding roots of monomials. For example, $12 = 2^2 \cdot 3$. $12^2 = 2^4 \cdot 3^2$. That is, if the factors of a number are known, the square root is found by taking for a product each factor half as many times as it occurs in the given number.

Ex. 1. Find square root of 625.

$$625 = 5^4.$$

Hence,
$$\sqrt{625} = \sqrt{5^4} = 5^2.$$

Ex. 2. Find cube root of 27.

$$\sqrt[3]{27} = \sqrt[3]{3^3} = 3.$$

Ex. 3. $\left(\dfrac{3x^2y}{4ab^3}\right)^3 = \dfrac{3^3(x^2)^3y^3}{4^3a^3(b^3)^3} = \dfrac{27x^6y^3}{64a^3b^9}.$

EXERCISE 75

Find the values of the following:

1. $\left(\dfrac{2a}{3}\right)^4.$
2. $\left(\dfrac{3a^2b}{7c}\right)^3.$
3. $\left(\dfrac{a^2b^3c^4}{9}\right)^2.$
4. $\left(\dfrac{15ax^4}{16d^7}\right)^2.$
5. $\sqrt{576}.$
6. $\sqrt[3]{512}.$
7. $(\sqrt{4a^2})^2.$
8. $(\sqrt[3]{8})^6.$
9. $(\sqrt[3]{8})^3.$
10. $(\sqrt[3]{64})^2.$
11. $(\sqrt{a^4b^6c^2})^3.$
12. $(\sqrt{64})^3.$
13. $(\sqrt[3]{343})^2.$
14. $\sqrt{1296}.$

$$1296 = 2^4 \cdot 3^4.$$

$$\therefore \sqrt{1296} = \sqrt{2^4 \cdot 3^4} = 2^2 \cdot 3^2 = 36.$$

15. $\sqrt{2025a^2b^4}.$
16. $\sqrt{a^{2m}}.$
17. $\sqrt{a^{2m+4}}.$
18. $\sqrt{x^{2y+6}}.$
19. $\sqrt{x^{4y-2}}.$
20. $\sqrt[3]{-343a^6}.$
21. $\sqrt[3]{a^{3m}}.$
22. $\sqrt[3]{125x^{3a+6}}.$
23. $\sqrt{96 \cdot 72}.$
24. $\sqrt[3]{\dfrac{125x^6y^3}{64a^3z^9}}.$
25. $\sqrt{108 \cdot 125 \cdot 15}.$
26. $\sqrt{\dfrac{729a^4c^2}{676x^6}}.$

27. $\sqrt{(a^2+a-2)(a^2+5a+6)(a^2+2a-3)}$.

28. $\sqrt{147 \cdot 27 \cdot 16}$.

29. $\sqrt{(x^2+2x-35)(x^2-3x-10)(x^2+9x+14)}$.

30. $\sqrt{(25z^2+70z+49)(9z^2-30z+25)}$.

31. $\sqrt[3]{9a^2 \cdot 15a \cdot 25a^6}$.

32. $\sqrt[3]{(27x^3)^2}$.

$$\sqrt[3]{(27x^3)^2}=(\sqrt[3]{27x^3})^2=(3x)^2=9x^2.$$

33. $\sqrt[5]{32}$. **35.** $\sqrt[4]{(16x^8y^4)^5}$. **37.** $\sqrt[4]{(81x^{4m+8})^2}$.

34. $\sqrt[5]{(32)^3}$. **36.** $\sqrt[5]{(243x^5y^{10})^3}$.

38. Compare with example 37, $\sqrt{81x^{4m+8}}$.

39. $\sqrt{(x^2+4x+4)^3}$. **41.** $\sqrt[6]{(64)^7y^6}$.

40. $\sqrt{(9x^2+12x+4)^3}$. **42.** $\sqrt[3]{(-512x^6y^3)^2}$.

Polynomials

217. Find the square of $a+b+c$.

$$\begin{array}{l}
a+b+c \\
\underline{a+b+c} \\
a^2+ab+ac \\
\quad +ab \quad +b^2+bc \\
\quad\quad \underline{+ac \quad +bc+c^2} \\
a^2+2ab+2ac+b^2+2bc+c^2
\end{array}$$

Or, $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc.$

Rule. *The square of any polynomial is equal to the sum of the squares of its terms, together with twice the product of each term by each term following it.*

Ex. 1. Expand $(2x^2-5x+3)^2$.

By the rule:

$$\begin{aligned}(2x^2-5x+3)^2 &= (2x^2)^2+(-5x)^2+(3)^2+2(2x^2)(-5x)+\\ &\qquad 2(2x^2)(3)+2(-5x)(3)\\ &=4x^4+25x^2+9-20x^3+12x^2-30x\\ &=4x^4-20x^3+37x^2-30x+9.\end{aligned}$$

Note the forms the terms of the answer were in before they were expanded.

Ex. 2. Extract the square root of

$$x^2 + 81\,y^2 + 16 + 18\,xy - 8\,x - 72\,y.$$

Expressing the factors of each term, as they appeared in the expansion of example 1,

$$(x)^2 + (9\,y)^2 + (4)^2 + 2(x)(9\,y) + 2(x)(-4) + 2(9\,y)(-4).$$

The square root is then $x + 9\,y - 4$.

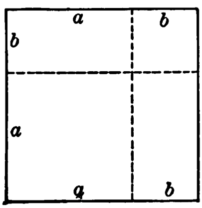
EXERCISE 76

Expand

1. $(x + y + 2)^2$.
2. $(x + y - z)^2$.
3. $(x + y - 2\,z)^2$.
4. $(x - 2\,y + z)^2$.
5. $(2\,a - b - c)^2$.
6. $(3\,a - 2\,b + 5\,c)^2$.
7. $(3\,a^2 - 2\,a + 5)^2$.
8. $(m^2 + 2\,m + 1)^2$.
9. $(5\,c^2 + 2\,c - 3)^2$.
10. $(8\,a + b + c)^2$.
11. $(2\,x^2 - 5\,x - 5)^2$.
12. $(3\,z - 2\,v + w)^2$.
13. $(6\,x - 14)^2$.
14. $(5\,y^2 - 3\,y + 2\,z)^2$.
15. $(x^3 - x^2 + 2\,x + 2)^2$.

Extract the square root:

16. $a^2 + b^2 + c^2 + 2\,ab - 2\,ac - 2\,bc$.
17. $4\,a^2 + 9\,b^2 + c^2 + 12\,ab - 4\,ac - 6\,bc$.
18. $x^2 + 4\,y^2 + 9\,z^2 - 4\,xy + 6\,xz - 12\,yz$.
19. $25\,x^2 + 4\,y^2 + 9\,z^2 - 20\,xy - 30\,xz + 12\,yz$.
20. $49\,m^2 + 16\,k^2 + 9\,t^2 + 56\,mk - 42\,mt - 24\,kt$.
21. $16\,a^4 + 25\,b^4 + 49\,d^2 - 40\,a^2b^2 - 56\,a^2d + 70\,b^2d$.
22. $25\,z^2 + 49\,k^2 + 81 + 70\,zk + 90\,z + 126\,k$.



218. Draw a line equal to the sum of two lines a and b. On this line build a square, erecting perpendiculars at each extremity of lines a and b. Note that now the square is made up of a^2, two rectangles, each $a \cdot b$, and b^2.

This fact helps us to formulate the rule for square roots.

Ex. Extract the square root of $a^2 + 2ab + b^2$.

Take out the greatest square possible, namely, a^2. The remainder, $2ab + b^2$, is made up of the two rectangles, ab, and the square b^2, on two sides of a^2.

The combined lengths of these rectangles is $2a$, and width b. If their area is divided by *twice* the length, or $2a$, the quotient is b, the width of the strip formed on two sides of the square a.

Hence,

1. *Extract the square root of the first term and write this for the first term of the root. Subtract the square of this number from the expression.*

2. *Double the root already found and divide into the remainder.* (This divisor is the trial divisor.)

3. *Add this quotient found to both root and trial divisor.* (This divisor is the complete divisor.)

4. *Multiply this complete divisor by the last number found in the root and subtract the product from the remainder.*

5. *If there is still a remainder, continue in the same manner until the square root is completed.*

EXERCISE 77

Extract the square root of:

1. $x^4 - 4x^3 + 10x^2 - 12x + 9$.
2. $4x^4 + 12x^3 - 19x^2 - 42x + 49$.
3. $x^2 + 4y^2 + 9 + 4xy - 6x - 12y$.
4. $9a^4 - 6a^3 + 31a^2 - 10a + 25$.
5. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.
6. $c^4 - 12c^3 + 54c^2 - 108c + 81$.
7. $16y^4 - 160y^3 + 600y^2 - 1000y + 625$.
8. $1 - 6x + 13x^2 - 12x^3 + 4x^4$.
9. $a^6 + 2a^5 + 3a^4 - 4a^3 - 5a^2 - 6a + 9$.
10. $4 + 12a + 13a^2 + 18a^3 + 19a^4 + 6a^5 + 9a^6$.

11. $9a^2 + 4b^2 + 25c^2 + 12ab - 30ac - 20bc.$

12. $16x^6 + 24x^5 + 25x^4 + 20x^3 + 10x^2 + 4x + 1.$

13. 625.

Since 625 may be treated as a polynomial, $600 + 20 + 5$, we may use the same rule in both arithmetic and algebra.

$$\begin{array}{r|l} & 600 + 20 + 5 \;|\; \underline{20 + 5} \\ (20)^2 = & 400 \\ \hline 40 + 5 & 200 + 20 + 5 \\ & \underline{200 + 20 + 5} \end{array}$$

If one keeps the decimal composition of number in mind, it is not necessary to actually form the polynomial number.

14. 15625.

In example 1, we had five terms and three terms in the root. In example 12, there were seven terms and four terms in the root.

In 15625 we have 1′56′25.
Find the square root of 1,
Consider the one as ten's,
Double the root,
Annex the next figure, 2, to the trial divisor; also write it in the root.

$$\begin{array}{r|l} & 1'56'25 \;|\; \underline{125} \\ & 1 \\ \hline \underline{20 + 2} & 56 \\ & 44 \\ \hline \underline{240 + 5} & 1225 \\ & \underline{1225} \end{array}$$

Continue in this manner until the entire root is found.

15. 2209. **16.** 3844. **17.** 12769. **18.** 43681.
19. 28224. **20.** 32400. **21.** 19600. **22.** 36100.
23. 52900. **24.** 262.44. **25.** 2.4336. **26.** 1.2769.
27. 1.5376. **28.** .003844.

29. 5. to three decimal places. **30.** 6. **31.** 7.

32. 8.

$$8 = 2^3 = 2^2 \cdot 2,$$

$$\sqrt{2^2 \cdot 2} = \sqrt{2^2}\sqrt{2} = 2\sqrt{2} = 2(1.414) = 2.828.$$

33. 50. **34.** 32. **35.** 12. **36.** .3. **37.** .2.

38. .5. (Compare example 33.) **39.** $\frac{2}{9}$. **40.** $\frac{2}{7}$.

41. $\frac{64}{81}$. **42.** $\frac{4}{5}$. **43.** $\frac{7}{2}$. **44.** $\frac{7}{8}$. **45.** $\frac{5}{82}$.

CHAPTER XIII

Exponents, Surds, Irrational Equations

219. In previous chapters we have learned that to raise a number to a power is to use the number as a factor a required number of times. Up to this time the index of the power has been an integer, also the exponent of a number used as a base has been an integer (§ 214).

Thus, $$a^3 \text{ means } a \cdot a \cdot a,$$

$$(a^3)^2 \text{ means } a^3 \cdot a^3 = a^{3+3},$$

or, $$a^5 \cdot a^2 = a^{5+2} = a^7.$$

Sometimes our base does not have an integral exponent.

Thus, $$a^{\frac{1}{2}},\ a^{\frac{1}{3}},\ a^{\frac{1}{5}}.$$

Illustrations: Raise $a^{\frac{1}{2}}$ to the second power.

$$(a^{\frac{1}{2}})^2 \text{ means } a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} \text{ or } a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$$

$$(a^{\frac{1}{3}})^3 \text{ means } a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a.$$

$a^{\frac{1}{2}}$ then means the square root of a, or is simply another way of writing $\sqrt{a}$.

Similarly, $a^{\frac{1}{3}}$ means the cube root of a, or $\sqrt[3]{a}$.

And, $a^{\frac{1}{n}}$ means nth root of a, or $\sqrt[n]{a}$.

$a^{\frac{2}{3}}$ means the cube root of a^2, for $(a^{\frac{2}{3}})^3 = a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} \cdot a^{\frac{2}{3}} = a^{\frac{6}{3}} = a^2$.

This is analogous to the reasoning, that if $5 \cdot 5 \cdot 5$ is 125, 5 is called the cube root of 125.

We may therefore conclude that if an exponent is *fractional*, the numerator of the exponent indicates to what power the

base is to be raised; the denominator, what root is to be extracted.

It is immaterial whether involution or evolution is performed first.

Thus, $$(64)^{\frac{2}{3}} = \sqrt[3]{(64)^2}, \text{ or } (\sqrt[3]{64})^2.$$

The latter reading is easier, for $(\sqrt[3]{64})^2 = (4)^2 = 16$.

$(64)^{\frac{2}{3}}$ may also be written: $[(64)^{\frac{1}{3}}]^2$, which means extract the cube root of 64 and square the result.

EXERCISE 78

Express with radical signs:

1. $a^{\frac{2}{3}} b^{\frac{1}{4}}$.

$$a^{\frac{2}{3}} \cdot b^{\frac{1}{4}} = \sqrt[3]{a^2} \cdot \sqrt[4]{b}, \text{ or } \sqrt[3]{a^2}\,\sqrt[4]{b}.$$

2. $a^{\frac{5}{6}}$. 3. $a^{\frac{3}{2}}$. 4. $8^{\frac{2}{3}}$. 5. $16^{\frac{1}{4}}$. 6. $81^{\frac{1}{4}}$.

Express with fractional exponents:

7. $\sqrt[3]{2^2}$.
8. $\sqrt[4]{16}$.
9. $\sqrt{8}$.
10. $\sqrt[4]{(5)^3}$.
11. $\sqrt[n]{a^2}$.
12. $\sqrt[n]{a^c}$.
13. $\sqrt[2m]{a^x b}$.
14. $\sqrt[3]{c^2 d} \cdot \sqrt{a^3}$.
15. $\sqrt[n]{a^2 b^2} \cdot \sqrt[3]{c^4}$.
16. $\sqrt[3]{5} \cdot \sqrt{7}$.
17. $\sqrt{a}\,\sqrt[3]{b}\,\sqrt[4]{c}$.
18. $\sqrt[3]{x^2}\,\sqrt[5]{x^{4+c}}$.

220. Another new form is the *zero* exponent. Since in obtaining the product of a^3 and a^2 we unite their factors (§ 34, example 2),

$$a^3 \cdot a^2 = a^{3+2}.$$

Similarly, $$a^3 \cdot a^0 = a^{3+0}$$
$$= a^3.$$

Then, $$a^3 \cdot a^0 = a^3$$

Dividing both sides of the equation by a^3,

$$a^0 = \frac{a^3}{a^3} = 1.$$

That is, *any number having a zero exponent is equal to* **1.**

221. One other type of exponent is needed, the *negative exponent.*

$$\begin{aligned} a^{-3} \cdot a^3 &= a^{-3+3} \\ &= a^0 \\ &= 1 \quad \text{(By § 220.)} \end{aligned}$$

Then, $$a^{-3} \cdot a^3 = 1. \tag{1}$$

Dividing both members by a^3,

$$a^{-3} = \frac{1}{a^3}. \tag{2}$$

That is, a number having a negative exponent is equal to the reciprocal of this number with the same positive exponent.

Had we divided equation (1) by a^{-3}, we would have had

$$a^3 = \frac{1}{a^{-3}}. \tag{3}$$

Equations (2) and (3) show that a *factor* may be changed from the numerator to the denominator, or from denominator to numerator of a fraction, if the sign of its exponent is changed.

Ex. $$\frac{a^{-3}}{b^{-2}c^4} = \frac{1}{a^3b^{-2}c^4} = \frac{b^2}{a^3c^4}.$$

It is generally preferable to have the *exponents positive.*

222. Since these three new types of exponents are founded on the same principles as those already in use, operations involving the various kinds of exponents are governed by the same rules and principles used in integral exponents.

Thus,
$$a^5 \cdot a^{-3} = a^{5-3} = a^2,$$
$$\frac{a^5}{a^{-3}} = a^5 \div a^{-3} = a^{5-(-3)} = a^8,$$
$$(a^{-3})^5 = a^{-3 \cdot 5} = a^{-15} = \frac{1}{a^{15}},$$
$$\sqrt[5]{a^{-3}} = a^{-\frac{3}{5}}, \text{ etc.}$$

Ex. 1. Multiply $a - 2a^{-1} + 3$ by $2a^{-1} + 5$.

$$
\begin{array}{l}
a - 2a^{-1} + 3 \\
\underline{2a^{-1} + 5} \\
2 - 4a^{-2} + 6a^{-1} \\
\underline{5a \qquad\qquad - 10a^{-1} + 15} \\
5a + 17 - 4a^{-2} - 4a^{-1}
\end{array}
$$

Or arranged according to the descending powers of a,

$$
\begin{array}{l}
a + 3 - 2a^{-1} \\
\underline{5 + 2a^{-1}} \\
5a + 15 - 10a^{-1} \\
\underline{\quad + 2 + 6a^{-1} - 4a^{-2}} \\
5a + 17 - 4a^{-1} - 4a^{-2}
\end{array}
$$

This latter is equivalent to $5a + 17a^0 - 4a^{-1} - 4a^{-2}$.

Ex. 2. Multiply $2a^{\frac{2}{3}} - 3a^{\frac{1}{3}} + 5$ by $2a^{-\frac{2}{3}} - 3a^{-\frac{1}{3}} + 5$.

$$
\begin{array}{l}
2a^{\frac{2}{3}} - 3a^{\frac{1}{3}} + 5 \\
\underline{2a^{-\frac{2}{3}} - 3a^{-\frac{1}{3}} + 5} \\
4 - 6a^{-\frac{1}{3}} + 10a^{-\frac{2}{3}} \\
-6a^{\frac{1}{3}} + 9 - 15a^{-\frac{1}{3}} \\
\underline{10a^{\frac{2}{3}} - 15a^{\frac{1}{3}} + 25} \\
10a^{\frac{2}{3}} - 21a^{\frac{1}{3}} + 38 - 21a^{-\frac{1}{3}} + 10a^{-\frac{2}{3}}
\end{array}
$$

Ex. 3. $(2a^{-\frac{1}{3}} + 5b^{\frac{2}{3}})^2$.

$$(2a^{-\frac{1}{3}} + 5b^{\frac{2}{3}})^2 = 4a^{-\frac{2}{3}} + 20a^{-\frac{1}{3}}b^{\frac{2}{3}} + 25b^{\frac{4}{3}}.$$

(By type II, Chapter VIII.)

EXERCISE 79

Find the following:

1. $(5a^2 + 6 - 4a^{-2})5a^{-2}$.
2. $(7a^2b^{-4}c^{\frac{1}{3}})(-6a^{-2}b^5c^{\frac{2}{3}})$.
3. $(16x^4y^{-5})(16x^{-4}y^5)$.
4. $1472 \cdot a^0$.
5. $(25)(25)^0$.
6. $(48632)(7468)^0$.
7. $(25a^2b^{-7}c)(13a^{-4}b^2)^0$.
8. $729 \div 27^0$.
9. $729 \div 27^{\frac{1}{3}}$.
10. $729 \div 27^{\frac{2}{3}}$.
11. $7.29 \div (.729)^0$.
12. $(841)^3(841)^{-3}$.
13. $(7a^2b)(7^{-1}a^{-2}b)$.
14. $18a^{-3}b^2 \div 9a^{-4}b$.

15. $(24\,a^{-4}b^2 - 36\,a^{-3}b^3 + 6\,a^{-2}b^4) \div 6\,a^{-3}b^3.$

16. $\dfrac{25\,x^4y^{-1} - 30\,x^3y^{-2} - 15\,x^0y^0}{5\,x^4y^{-1}}.$

17. $\dfrac{(2a+b)^2(2a+b)^0 - (2a+b)^3(2a+b)^{-1} - (2a+b)^4(2a+b)^{-2}}{(2a+b)^2}.$

18. $\dfrac{(3x+2y)^2 - (3x+2y) - 1}{(3x+2y)^0}.$

19. $(x^{-1} - 9)^2.$

20. $(x^{-1} - 9)(x^{-1} + 11).$

21. $(2\,x^{\frac{1}{3}} + 5\,y^{\frac{2}{3}})(2\,x^{\frac{1}{3}} - 5\,y^{\frac{2}{3}}).$

22. $\dfrac{4\,a^{-4} - 12\,a^{-2} + 9\,a^0}{2\,a^{-2} - 3}.$

23. Factor $m^{-2} - 8\,m^{-1} + 15.$

24. Factor $16\,x^{-\frac{4}{5}} - 25\,y^{-\frac{4}{3}}.$

25. Factor $16\,x^{-\frac{4}{5}} - 25\,y^{-3}.$

26. Factor $9\,a - 16\,b$. (Call a and b squares.)

27. $(x^{-2} + x^{-1} + 1)(x^{-1} - 1).$

28. $(a^{-1} + b^{-1})^2.$

29. $x + 2\,x^{\frac{1}{2}} +$? is a square?

Fill in the missing term in the following trinomial squares:

30. $x + 6\,x^{\frac{1}{2}} + ?$

31. $a^{-4} - 8\,a^{-2} + ?$

32. $x^{-4} + 4\,x^{-2} + ?$

33. $x^{-4} - ? + 625.$

34. $16\,m^{-4} + 625\,m^0 + ?$

35. $9\,c^{-2} + 12\,c^{-1} + ?$

36. $a^{-4} + 4\,a^0.$

37. $x^{-4} + (\quad) + 81.$

38. $144\,b^{-2} + (\qquad) + a^{-4}.$

Factor:

39. $x^{-2} + 4\,x^{-1}y - 21\,y^2.$

40. $a^{-3} - 5\,a^{-2}z - 24\,a^{-1}z^2.$

41. $z^{-4} - 81.$

42. $(a+b)^{-2} + (a+b)^{-1} - 72(a+b)^0.$

43. $(a-y)^{-2} - (a-y)^{-1} - 56(a-y)^0.$

44. $a^{-4} - 50\,a^{-2} + 49.$

45. $16\,x^{\frac{4}{3}} - 40\,x^{\frac{2}{3}} + 25.$

46. $25\,a - 10\,a^{\frac{1}{2}} + 1.$ (Call $25\,a$ a square.)

47. $(x^{-2} - 4\,b^{-3})(x^{-2} + 9\,b^{-3}) = ?$

48. $(2\,a^{-3} + 5\,b^{-3})(2\,a^{-3} + 8\,b^{-3})$.

49. $(25\,a - 36\,b) \div (5\,a^{\frac{1}{2}} + 6\,b^{\frac{1}{2}})$.

50. $(x - 1) \div (x^{\frac{1}{4}} - 1)$.

51. $(a - 5\,a^{\frac{2}{3}}m^{\frac{1}{6}} + 9\,a^{\frac{1}{3}}m^{\frac{1}{3}} - 9\,m^{\frac{1}{2}}) \div (a^{\frac{2}{3}} - 2\,a^{\frac{1}{3}}m^{\frac{1}{6}} + 3\,m^{\frac{1}{3}})$.

52. $25^{-\frac{1}{2}} = ?$ Express decimally.

53. $\dfrac{x^{-1} + 2\,y}{y^{-1}} = ?$

$$= \frac{\frac{1}{x} + 2\,y}{y^{-1}} = \left(\frac{1}{x} + 2\,y\right)y$$

$$= \frac{y + 2\,xy^2}{x}.$$

54. $\sqrt{x - 2\,x^{\frac{1}{2}}y^{\frac{1}{2}} + y} = ?$

55. $(x^2 - x^{-2})^2 + 4 = ?$

56. $1 + \left(\dfrac{a^5 - a^{-5}}{2}\right)^2 = ?$

57. $\left[\dfrac{\left(\dfrac{a^5 + a^{-5}}{2}\right)^2 - 1}{(a^5 - a^{-5})^2}\right]4 = ?$

58. $\dfrac{a^{\frac{1}{3}} + b^{\frac{1}{3}}}{a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}} + \dfrac{a^{\frac{1}{3}} - b^{\frac{1}{3}}}{a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}} = ?$

59. $(3^{a+5} - 3^a \cdot 3)(3^{-1} \cdot 3^{-a}) = ?$

60. $(5^{x+6} - 5^x \cdot 5^4)(5^{-6} \cdot 5^{-x})$. *Ans.* $\frac{24}{25}$.

61. Is $a^{\frac{1}{2}} + 10\,a^{\frac{1}{4}}b^{\frac{1}{4}} + 25\,b^{\frac{1}{2}}$ a square?

62. $(5^{\frac{1}{2}} + 12^{\frac{1}{2}})^2 = ?$

63. Is $5 + 2\sqrt{35} + 7$ a square?

64. Can you factor $6 - 2\sqrt{48} + 8$?

65. Express with positive exponents: $\dfrac{(a+b)^{-\frac{1}{2}}}{(a-b)^{\frac{1}{2}}}$.

66. Express with positive exponents: $\dfrac{a^{-1}+b^{-1}}{a^{-1}-b^{-1}}$.

67. $\left(x^{a-\frac{b^2}{a}}\right)^{\frac{a}{a-b}}$.

68. Extract the square root:

$$4\,m^{-4}+12\,m^{-3}-7\,m^{-2}-24\,m^{-1}+16.$$

69. Extract the square root:

$$c^2-2\,c^{\frac{5}{3}}-c^{\frac{4}{3}}+6\,c-3\,c^{\frac{2}{3}}-4\,c^{\frac{1}{3}}+4.$$

70. Extract the square root:

$$9\,m-12\,m^{\frac{5}{6}}+28\,m^{\frac{2}{3}}-22\,m^{\frac{1}{2}}+20\,m^{\frac{1}{3}}-8\,m^{\frac{1}{6}}+1.$$

71. Extract the square root:

$$x^{2n}+12\,x^{\frac{3n}{2}}+54\,x^n+108\,x^{\frac{n}{2}}+81.$$

72. $(5\cdot 2^{\frac{1}{2}}+3\cdot 5^{\frac{1}{2}})^2$. **73.** $(5+2^{\frac{1}{2}})^2$. **74.** $(5^{\frac{1}{2}}-3^{\frac{1}{2}})^2$.

75. Extract the square root: $4\,x+4\,x^{\frac{1}{2}}+1$.

76. Extract the square root: $3+2\sqrt{21}+7$.

77. Multiply $5\,a^{-1}+6\,a^{-2}+7$ by $3-5\,a^{-2}+6\,a^{-1}$.

78. Multiply $5\,a^2+6\,a+7$ by $3-5\,a^{-2}+6\,a^{-1}$.

79. Multiply $5\,(x-y)^{-2}+6\,(x-y)^{-1}+8$ by $2\,(x-y)^{-1}+7$.

80. Factor $64\,x^{\frac{1}{2}}(8\,x^{\frac{1}{4}}+9)+144\,x^{\frac{1}{4}}(8\,x^{\frac{1}{4}}+9)+81\,(8\,x^{\frac{1}{4}}+9)$.

81. $(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})=?$

82. $(3^{\frac{1}{2}}+2^{\frac{1}{2}})(3^{\frac{1}{2}}-2^{\frac{1}{2}})=?$ **83.** $(2\,a^{\frac{1}{2}}+7\,b^{\frac{1}{2}})(2\,a^{\frac{1}{2}}-7\,b^{\frac{1}{2}})$.

84. $[(2\,a)^{\frac{1}{2}}+(7\,b)^{\frac{1}{2}}][(2\,a)^{\frac{1}{2}}-(7\,b)^{\frac{1}{2}}]$.

223. The indicated root of a number which is not a perfect power of the degree indicated by the index of the root is a **surd.**

That is, the square root of any arithmetical number not belonging to the set 1, 4, 9, 16, 25, 36, 49, etc., is a **quadratic surd.** The cube root of any number not belonging to the set 1, 8, 27, 64, 125, etc., is a **cubic surd.**

224. Any expression involving surds is **irrational.** A **rational expression** contains no surds.

E.g. $\frac{2 + 10\,y}{7\,c}$ is rational. $2 + \sqrt{3}$ and $\frac{5\,x + \sqrt{y}}{7\sqrt{c}}$ are irrational.

To be in its *simplest form* a *surd* must be an integer and must contain no factor which is a perfect power of the degree indicated by the index of the root.

E.g. $\sqrt{8}$ is not in its simplest form, for

$\sqrt{8} = \sqrt{4 \cdot 2}$. 4 is a square, so the surd is not in its simplest form. This number is evidently the product of $\sqrt{4}$ and $\sqrt{2}$, or $4^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 2 \cdot 2^{\frac{1}{2}}$. Note how naturally the reduction takes place if we think of 4 as 2^2. We then have

$$(2^2)^{\frac{1}{2}} \cdot (2)^{\frac{1}{2}} = 2^{\frac{2}{2}} \cdot (2)^{\frac{1}{2}} = 2 \cdot 2^{\frac{1}{2}}.$$

Ex. 1. Simplify $\sqrt{48}$.

$$(48)^{\frac{1}{2}} = (2^4 \cdot 3)^{\frac{1}{2}} = 2^{\frac{4}{2}} \cdot 3^{\frac{1}{2}} = 2^2 \cdot 3^{\frac{1}{2}} = 4 \cdot 3^{\frac{1}{2}} \text{ or } 4\sqrt{3}.$$

Ex. 2. Simplify $\sqrt[6]{27}$.

$$\sqrt[6]{27} = 27^{\frac{1}{6}} = 27^{\frac{1}{3} \cdot \frac{1}{2}} = (27^{\frac{1}{3}})^{\frac{1}{2}} = 3^{\frac{1}{2}}.$$

Solving by factors,

$$\sqrt[6]{27} = \sqrt[6]{3^3} = 3^{\frac{3}{6}} = 3^{\frac{1}{2}}.$$

EXERCISE 80

Simplify:

1. $\sqrt[6]{8}$. **3.** $\sqrt{27}$. **5.** $(54)^{\frac{1}{3}}$. **7.** $(100)^{\frac{1}{4}}$.

2. $\sqrt[6]{4}$. **4.** $(50)^{\frac{1}{2}}$. **6.** $(25)^{\frac{1}{4}}$. **8.** $(216\,x^3y^6)^{\frac{1}{6}}$.

9. $(a^3 + 6\,a^2 + 9\,a)^{\frac{1}{2}}$. **12.** $(32)^{\frac{1}{10}}$.

10. $[(4\,x^2 - 12\,xy + 9\,y^2)y]^{\frac{1}{2}}$. **13.** $\sqrt[6]{2^7}$.

11. $(16\,c^2 - 40\,cd + 25\,d^2)^{\frac{1}{4}}$.

14. $[(2\,x + 3\,y)4\,x^2 + (2\,x + 3\,y)12\,xy + (2\,x + 3\,y)9\,y^2]^{\frac{1}{2}}$.

15. $(512\,a^3b)^{\frac{1}{2}}$. **17.** $[(a^2 + 2\,ab + 4\,b^2)a^2]^{\frac{1}{2}}$.

16. $(9\,a^2 + 42\,ab + 49\,b^2)^{\frac{1}{4}}$. **18.** $[(a^2 + 4\,ab + 4\,b^2)a]^{\frac{1}{2}}$.

19. $[(x-4)(x^2-16)]^{\frac{1}{2}}$.

20. $[(x+3)(x^2+7x+12)]^{\frac{1}{2}}$.

21. $\sqrt[3]{\frac{5}{9}}$.

$$\left(\frac{5}{9}\right)^{\frac{1}{3}}=\left(\frac{5}{3^2}\right)^{\frac{1}{3}}=\left(\frac{5\cdot 3}{3^3}\right)^{\frac{1}{3}}=\frac{(15)^{\frac{1}{3}}}{(3^3)^{\frac{1}{3}}}=\frac{1}{3}\sqrt[3]{15}.\quad (\S\ 224.)$$

22. $(\frac{7}{12})^{\frac{1}{2}}$. $\left(\textit{Hint.}\ \frac{7}{12}=\frac{7}{2^2\cdot 3}\cdot\right)$

23. $(\frac{175}{12})^{\frac{1}{2}}$.

24. $(\frac{7}{8})^{\frac{1}{2}}$.

25. $(\frac{5}{28})^{\frac{1}{2}}$.

26. $(\frac{11}{40})^{\frac{1}{2}}$.

27. $(\frac{11}{40})^{\frac{1}{3}}$.

28. $(\frac{7}{75})^{\frac{1}{2}}$.

29. $(\frac{9}{25})^{\frac{1}{3}}$.

30. $\left(\frac{35x^2}{48a^3b}\right)^{\frac{1}{2}}$.

31. $\left(\frac{a-b}{a}\right)^{\frac{1}{2}}$.

32. $\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}}$.

33. $\left[\frac{a-b}{(a+b)^2}\right]^{\frac{1}{3}}$.

34. $\left(\frac{a-b}{2a^2+4ab+2b^2}\right)^{\frac{1}{2}}$.

35. $\left[\left(\frac{a-b}{a+b}\right)-1\right]^{\frac{1}{2}}$.

36. $\left[\left(\frac{a-b}{2}\right)^2+ab\right]^{\frac{1}{2}}$.

37. $\left[\frac{1}{2x^2(x+y)^2}\right]^{\frac{1}{3}}$.

38. $\left[\frac{5}{3a(a+3)^2}\right]^{\frac{1}{2}}$.

39. $\left[\frac{6(a+7)}{5a^2(a-7)}\right]^{\frac{1}{2}}$.

40. $\left[\frac{\sqrt{2}+\sqrt{3}}{6x(x+2)}\right]^{\frac{1}{2}}$.

In examples 20–40, the *entire* denominator was affected by the fractional exponent. It was therefore necessary to multiply both numerator and denominator by some number that would make the denominator a perfect power. Sometimes the *terms* of the denominator are affected by the fractional exponent. It is then necessary to resort to the principle in § 153 to find the required multiplier.

E.g. Simplify $\frac{2}{\sqrt{3}+\sqrt{2}}\cdot$ (See example 82, exercise 79.)

$$\frac{2}{3^{\frac{1}{2}}+2^{\frac{1}{2}}}=\frac{2(3^{\frac{1}{2}}-2^{\frac{1}{2}})}{(3^{\frac{1}{2}}+2^{\frac{1}{2}})(3^{\frac{1}{2}}-2^{\frac{1}{2}})}=\frac{2\sqrt{3}-2\sqrt{2}}{3-2}=2\sqrt{3}-2\sqrt{2}.$$

$3^{\frac{1}{2}}+2^{\frac{1}{2}}$ and $3^{\frac{1}{2}}-2^{\frac{1}{2}}$ are called *conjugate binomials*.

Simplify:

41. $\dfrac{3a^{\frac{1}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$.

42. $\dfrac{6}{\sqrt{5}-\sqrt{2}}$.

43. $\dfrac{4}{\sqrt{5}+2}$.

44. $\dfrac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}$.

45. $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}$.

46. $\dfrac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$.

47. $\dfrac{8\sqrt{2}}{5\sqrt{2}-\sqrt{4}}$.

48. $\dfrac{\sqrt{7}+2}{\sqrt{7}+3}$.

49. $\dfrac{\sqrt{7}+2}{\sqrt{7}-2}$.

50. $\dfrac{1-\sqrt{2}}{1+\sqrt{2}}$.

51. $\dfrac{1+a^{\frac{1}{2}}}{1-a^{\frac{1}{2}}}$.

52. $\dfrac{x+2\sqrt{y}}{x-2\sqrt{y}}$.

53. $\dfrac{x+(2y)^{\frac{1}{2}}}{x-(2y)^{\frac{1}{2}}}$.

54. $\left(\frac{18}{7}\right)^{\frac{1}{2}}$.

55. $\left(\frac{4}{15}\right)^{\frac{1}{2}}$.

56. $\left(\frac{8}{9}\right)^{\frac{1}{3}}$.

57. $\left[\dfrac{(a+b)^2}{a-b}\right]^{\frac{1}{2}}$.

58. $\left[\dfrac{(2x+3)^3}{(2x-3)^2}\right]^{\frac{1}{3}}$.

59. $\dfrac{\sqrt{2x}+\sqrt{3}}{\sqrt{2x}-\sqrt{3}}$.

60. $\left(\dfrac{50x^2y}{27a}\right)^{\frac{1}{2}}$.

61. $\dfrac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}$.

In examples 21–61 we made the denominators rational. Occasionally it is necessary to rationalize the numerator.

Simplify the numerator:

62. $\dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}-\sqrt{3}}$.

(Multiply both numerator and denominator by the conjugate of the numerator.)

$$\frac{2^{\frac{1}{2}}+3^{\frac{1}{2}}}{2^{\frac{1}{2}}-3^{\frac{1}{2}}}\cdot\frac{2^{\frac{1}{2}}-3^{\frac{1}{2}}}{2^{\frac{1}{2}}-3^{\frac{1}{2}}}=\frac{2-3}{2-2\cdot 6^{\frac{1}{2}}+3}=-\frac{1}{5-2\cdot 6^{\frac{1}{2}}}.$$

Rationalize the numerators in the following:

63. $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$.

64. $\dfrac{\sqrt{x^2-1}+\sqrt{x^2+1}}{\sqrt{x^2-1}-\sqrt{x^2+1}}$.

65. $\sqrt{5}$.

$$\sqrt{5}=\frac{\sqrt{5}}{1}=\frac{\sqrt{5}}{1}\cdot\frac{\sqrt{5}}{\sqrt{5}}=\frac{5}{\sqrt{5}}.$$

66. $2^{\frac{1}{2}}$.

67. $3\cdot 3^{\frac{1}{2}}$.

68. $\sqrt{2\,ax-x^2}$.

69. $\sqrt{1-3\,x-x^2}$.

70. $a^{\frac{1}{2}}+b^{\frac{1}{2}}$.

225. In exercise 80, we simplified each irrational number. This simply meant the number was separated into two factors, one of which was a perfect power of the degree shown by the index of the root. (See example 1, § 224.) It is often necessary to express a term as a single irrational factor. That is, we must make the coefficient of the irrational part also irrational. This is merely finding the product of the irrational and the rational parts of the term.

E.g. express $2\sqrt{3}$ as a single irrational factor. $2\cdot 3^{\frac{1}{2}}$ is to be expressed as one factor. This cannot be done until the exponents have the same denomination.

$$2\cdot 3^{\frac{1}{2}}=2^{\frac{2}{2}}\cdot 3^{\frac{1}{2}}=(2^2\cdot 3^1)^{\frac{1}{2}}=(12)^{\frac{1}{2}},\text{ or }\sqrt{12}.$$

This is called *introducing the coefficient under the radical sign.* Similarly in finding the product of irrational numbers of different degree, one must reduce the exponents to the same denomination.

Thus,

$$2^{\frac{1}{2}}\cdot 3^{\frac{1}{3}}=2^{\frac{3}{6}}\cdot 3^{\frac{2}{6}}=(2^3\cdot 3^2)^{\frac{1}{6}}=\sqrt[6]{72}.$$

EXERCISE 81

Introduce the coefficient under the radical sign:

(Make the expression wholly irrational.)

1. $15\sqrt{5}$.

$$15\cdot 5^{\frac{1}{2}}=15^{\frac{2}{2}}\cdot 5^{\frac{1}{2}}=(15^2\cdot 5)^{\frac{1}{2}}=(5^3\cdot 3^2)^{\frac{1}{2}}=\sqrt{1125}.$$

2. $4\sqrt{7}$. 4. $3\cdot 3^{\frac{1}{2}}$. 6. $8\sqrt[3]{5}$. 8. $2\sqrt[4]{2}$. 10. $2\cdot 3^{\frac{1}{3}}$.

3. $5\cdot 5^{\frac{1}{2}}$. 5. $8\sqrt{8}$. 7. $3\cdot 5^{\frac{1}{3}}$. 9. $3\cdot 3^{\frac{1}{4}}$. 11. $2\cdot 2^{\frac{1}{10}}$.

12. $4\sqrt[3]{48}$.

13. $(a+b)\sqrt{a+b}$.

14. $\frac{1}{5}\sqrt{5}$.

15. $\frac{2}{3}\sqrt{7}$.

16. $\frac{2}{3}\cdot 3^{\frac{1}{3}}$.

17. $\frac{3}{5}\cdot 5^{\frac{1}{2}}$.

18. $\dfrac{1}{a+b}\cdot(a+b)^{\frac{1}{2}}$.

19. $\dfrac{a-b}{a+b}\sqrt{\dfrac{a+b}{a-b}}$.

20. $\dfrac{2}{2x-y}\left(\dfrac{2x-y}{8}\right)^{\frac{1}{2}}$.

21. $(a-2b)\left(\dfrac{1}{a-2b}\right)^{\frac{1}{2}}$.

22. $(a-2b)\left[\dfrac{1}{(a-2b)^2}\right]^{\frac{1}{2}}$.

23. $\dfrac{2x-3y}{2x+3y}\left(\dfrac{2x+3y}{2x-3y}\right)^{\frac{1}{2}}$.

24. $(4a^2-b^2)\left[\dfrac{2a-b}{(4a^2-b^2)(2a+b)}\right]^{\frac{1}{2}}$.

Express in such a manner that 2 will be the coefficient of the irrational part: $6\sqrt{15}$.

$$6\sqrt{15}=6\cdot 15^{\frac{1}{2}}=2\cdot 3\cdot 15^{\frac{1}{2}}=2\cdot 3^{\frac{2}{2}}\cdot 15^{\frac{1}{2}}=2(3^2\cdot 15)^{\frac{1}{2}}=2(135)^{\frac{1}{2}}.$$

25. $8\sqrt{3}$. **26.** $12\sqrt{2}$.

27. $\sqrt{24}$.

$$24^{\frac{1}{2}}=(2^3\cdot 3)^{\frac{1}{2}}=2^{\frac{2}{2}}\cdot 2^{\frac{1}{2}}\cdot 3^{\frac{1}{2}}=2\sqrt{6}.$$

28. $48^{\frac{1}{2}}$. **30.** $32^{\frac{1}{2}}$. **32.** $\sqrt{20}$. **34.** $\sqrt{52}$. **36.** $8\cdot 7^{\frac{1}{2}}$.

29. $28^{\frac{1}{2}}$. **31.** $\sqrt{40}$. **33.** $\sqrt{44}$. **35.** $56^{\frac{1}{2}}$. **37.** $10\cdot 5^{\frac{1}{2}}$.

38. $5\cdot 3^{\frac{1}{2}}$.

$$5\cdot 3^{\frac{1}{2}}=2\cdot\tfrac{5}{2}\cdot 3^{\frac{1}{2}}=2\cdot(\tfrac{5}{2})^{\frac{2}{2}}(3)^{\frac{1}{2}}=2(\tfrac{75}{4})^{\frac{1}{2}}=2\sqrt{\tfrac{75}{4}}.$$

39. $\sqrt{15}$.

$$15^{\frac{1}{2}}=4^{\frac{1}{2}}\cdot(\tfrac{15}{4})^{\frac{1}{2}}=2\sqrt{\tfrac{15}{4}}.$$

(Note that the factor to be removed is 4 and not 2.)

40. $5\sqrt{48}$. **41.** $\sqrt{17}$. **42.** $7\cdot 28^{\frac{1}{2}}$.

43. $3\cdot 15^{\frac{1}{2}}$.

$$3\cdot 15^{\frac{1}{2}}=2\cdot\tfrac{3}{2}\cdot 15^{\frac{1}{2}}=2\cdot(\tfrac{3}{2})^{\frac{2}{2}}\cdot 15^{\frac{1}{2}}=2\sqrt{\tfrac{135}{4}}.$$

44. $(135)^{\frac{1}{2}}$.

45. $3\sqrt{19}$.

46. $5\sqrt{18}$.

47. $(2a+2b)(a+b)^{\frac{1}{2}}$.

48. $\sqrt{4x^2-8xy+24y^2}$.

49. $3x-3z\sqrt{7}$.

50. $6\sqrt{a^2+3ab+9b^2}$.

51. $\dfrac{1}{3x-2y^2}\sqrt{9x^2-4y^4}$.

52. $\dfrac{a+b}{a-b}\sqrt{\dfrac{a-b}{a+b}}$.

53. $\sqrt[12]{625x^{24}y^6}=?$

226. Irrational numbers are added and subtracted in the same manner as rational numbers.

Similar irrational numbers may differ *only* in their coefficients. The coefficients of similar irrational numbers may be united by addition or subtraction.

Ex. Find the sum of $\sqrt{12}$ and $\sqrt{48}$.

$$12^{\frac{1}{2}}=2\cdot 3^{\frac{1}{2}}$$
$$48^{\frac{1}{2}}=4\cdot 3^{\frac{1}{2}}$$
$$12^{\frac{1}{2}}+48^{\frac{1}{2}}=6\cdot 3^{\frac{1}{2}}$$

EXERCISE 82

Simplify:

1. $\sqrt{50}+\sqrt{32}-\sqrt{12}$.

2. $75^{\frac{1}{2}}-27^{\frac{1}{2}}$.

3. $2\cdot 2^{\frac{1}{2}}-50^{\frac{1}{2}}-98^{\frac{1}{2}}$.

4. $40^{\frac{1}{3}}-135^{\frac{1}{3}}+320^{\frac{1}{3}}$.

5. $\sqrt{5}+\sqrt{20}+\sqrt{45}$.

6. $\sqrt{(a+b)^3}-5a\sqrt{a+b}+3b\sqrt{a+b}$.

7. $\sqrt{(x+y)^2(x-y)}-\sqrt{9x^3-9x^2y}$.

8. $(25a^4-100a^2b^2)^{\frac{1}{2}}-\sqrt{16a^2-64b^2}$.

9. $(25a^4-100a^2b^2)^{\frac{1}{2}}-(16a^4-64a^2b^2)^{\frac{1}{2}}$.

10. $(3a^2x-18ax+27x)^{\frac{1}{2}}-(3a^2x-24ax+48x)^{\frac{1}{2}}$.

11. $3(50\,yz^2 - 60\,yzk + 18\,yk^2)^{\frac{1}{2}} - 2\,(9\,yz^2 - 30\,yzk + 25\,yk^2)^{\frac{1}{2}}$.

12. $\sqrt{80\,x^3 + 40\,x^2y + 5\,xy^2} + \sqrt{20\,x^3 - 60\,x^2y + 45\,xy^2}$.

13. $\sqrt{\frac{9}{8}} + \sqrt{\frac{25}{2}}$

$$= \tfrac{3}{4}\sqrt{2} + \tfrac{5}{2}\sqrt{2} = \tfrac{13}{4}\sqrt{2}.$$

14. $\sqrt{\frac{27}{4}} - \sqrt{\frac{3}{4}}$.

15. $20^{\frac{1}{2}} - (\frac{9}{5})^{\frac{1}{2}}$.

16. $(\frac{8}{9})^{\frac{1}{3}} + (\frac{81}{8})^{\frac{1}{3}}$.

17. $\sqrt{42} - \sqrt{243} + \sqrt{128}$.

18. $\sqrt[3]{16} - \sqrt[4]{32}$.

19. $\sqrt[5]{a^4} - \sqrt[4]{a^5}$.

20. $\dfrac{(108)^{\frac{1}{2}}}{3} + \dfrac{(192)^{\frac{1}{2}}}{5} - \dfrac{(432)^{\frac{1}{2}}}{2}$.

21. $\dfrac{\sqrt{245}}{7} - \dfrac{\sqrt{125}}{5} + \dfrac{(180)^{\frac{1}{2}}}{6}$.

22. $\sqrt{\dfrac{(x^2 + x - 6)(x + 3)}{(x^2 - x - 6)(x - 3)}} + \sqrt{\dfrac{(x^2 + 3\,x - 10)(x + 5)}{16\,x + 32}}$.

23. $\dfrac{\sqrt[3]{24}}{2} + \dfrac{\sqrt[3]{81}}{3} + \dfrac{\sqrt[3]{192}}{4}$.

24. $\sqrt[3]{\frac{8}{9}} - (\frac{1}{72})^{\frac{1}{3}} + (\frac{375}{512})^{\frac{1}{3}}$.

25. $\sqrt[6]{144\,x^6y^3} = ?$

26. $\frac{3}{7}\sqrt[3]{3} + \sqrt[3]{\frac{125}{73}} + 32\sqrt[3]{\frac{1}{27}}$.

227. To find the value of an expression containing irrational numbers in the denominator, much labor is saved by rationalizing the denominator.

Ex. Find the value of $\dfrac{1}{\sqrt{2}}$.

$\sqrt{2} = 1.414 +$.

Hence, $\dfrac{1}{\sqrt{2}} = \dfrac{1}{1.414} = 1 \div 1.414 = .707+$.

Had we rationalized the denominator, the divisor would have been *small* and the value could have been found mentally. Thus,

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \frac{1.414}{2} = .707.$$

EXERCISE 83

Find values to three decimal places:

1. $\dfrac{1}{\sqrt{3}}$.

2. $\dfrac{1}{\sqrt{5}}$.

3. $\dfrac{2}{2-\sqrt{3}}$.

4. $\dfrac{3}{3+\sqrt{2}}$.

5. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}}$.

6. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

7. $\dfrac{4+2^{\frac{1}{2}}}{4-2^{\frac{1}{2}}}$.

8. $\dfrac{2\sqrt{3}}{\sqrt{5}+\sqrt{3}}$.

9. $\dfrac{1}{\sqrt{6}-2}$.

10. $\dfrac{3}{\sqrt{3}-2}$.

11. $\dfrac{5+\sqrt{2}}{5-\sqrt{2}}$.

12. $\dfrac{3-\sqrt{5}}{3+\sqrt{5}}$.

13. $\dfrac{\sqrt{7}+2}{\sqrt{7}-2}$.

14. $\dfrac{2\cdot 5^{\frac{1}{2}}+2}{2\cdot 5^{\frac{1}{2}}-2}$.

Add the following:

15. $\dfrac{1}{5+\sqrt{2}}+\dfrac{1}{5-\sqrt{2}}$.

16. $\dfrac{3}{\sqrt{3}+2}-\dfrac{3}{\sqrt{3}-2}$.

17. $\dfrac{x^{\frac{1}{2}}+y}{x^{\frac{1}{2}}-y}-\dfrac{x^{\frac{1}{2}}-y}{x^{\frac{1}{2}}+y}$.

18. $\dfrac{(a+1)^{\frac{1}{2}}+1}{(a+1)^{\frac{1}{2}}-1}-\dfrac{2(a+1)^{\frac{1}{2}}}{a}$.

19. $\dfrac{a^{-1}-b^{-1}}{a^{-\frac{1}{2}}+b^{-\frac{1}{2}}}+a^{-\frac{1}{2}}-b^{-\frac{1}{2}}$.

Express the result with positive exponents.

20. $\dfrac{1-\sqrt{\dfrac{1}{x}}}{1+\sqrt{\dfrac{1}{x}}}+\dfrac{\sqrt{x}+1}{\sqrt{x}-1}$.

21. $\dfrac{(x^2+a^2)^{\frac{1}{2}}+a}{(x^2+a^2)^{\frac{1}{2}}-a}+\dfrac{x^2+a^2}{x^2}$.

22. $\dfrac{a^{-\frac{1}{2}}+b^{-\frac{1}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}-\dfrac{1}{\sqrt{ab}}$.

23. $\dfrac{1}{2+\sqrt{2}}+\dfrac{1}{2-\sqrt{2}}$.

24. The projection of one leg of a right triangle upon the hypotenuse is 48. One angle is 60°. Find the angles, lines, and area of the triangle.

25. The projection of one leg of a right triangle upon the hypotenuse is $(x^2 + 6x + 9)\sqrt{3}$. One angle is $30°$. Find all lines and angles of the triangle. Find area. Compute the area when $x = 1$. Is there more than one solution?

26. In a right triangle *ABC*, the projection, *AD*, of *AC* upon the hypotenuse is $\sqrt{12x^2 + 48x + 48}$, the hypotenuse is twice the shorter side. Solve the triangle.

27. In a right triangle the projection of one leg upon the hypotenuse is $(27x^2 - 36x + 12)^{\frac{1}{2}}$. Solve the triangle when the hypotenuse is twice the shorter side.

28. Two parallels are cut by a transversal. The segment of the transversal between the parallels is 42. The angle which the transversal makes with one of the parallels is $120°$. Bisect all interior angles. Find the area of the quadrilateral formed.

29. One side of a square is 24. Find the diagonal.

30. The area of a square is 729. Find the diagonal.

31. The area of a square is $25x^2 - 80x + 64$. Find the diagonal.

32. The area of a square is $4x^{\frac{1}{2}} - 12x^{\frac{1}{4}} + 9$. Find the diagonal. Compute the area, side, and diagonal when $x = 16$.

33. The area of a square is $9x^{\frac{1}{3}} - 30x^{\frac{1}{6}} + 25$. Find the diagonal. How long is the diagonal when $x = 64$? Compare this square with the one in example 32.

34. The area of a square is $16x^{\frac{1}{5}} - 56x^{\frac{1}{10}} + 49$. Find the diagonal. Compute when $x = 1024$.

35. The area of a square is $36x^{\frac{1}{3}} + 12x^{\frac{1}{6}} + 1$. Find the diagonal. Compute when $x = \frac{1}{729}$.

36. The diagonal of a square is $(2x^{\frac{1}{10}} + 5)\sqrt{2}$. Find the area.

37. The diagonal of a square is $(2x^{\frac{1}{10}}-1)(2)^{\frac{1}{2}}$. Compute the area when $x=\frac{1}{1024}$.

38. The area of a square is $2+2\sqrt{6}+3$. Find one side.

39. The area of a square is $a+2a^{\frac{1}{2}}b^{\frac{1}{2}}+b$. Find one side.

40. $81^{.25}=?$ **42.** $.125^{.333}=?$ **44.** $.0625^{.25}=?$

41. $64^{.33333}=?$ **43.** $256^{.125}=?$ **45.** $1024^{.2}=?$

228. In exercise 77, we learned to extract the square roots of numbers. Many square roots may be found by inspection.

First commit to memory the square roots of 2, 3, and 5.

$$\sqrt{2}=1.414,\quad \sqrt{3}=1.732,\quad \sqrt{5}=2.236.$$

Ex. 1. Find $\sqrt{48}$.

$$\sqrt{48}=(2^4\cdot 3)^{\frac{1}{2}}=4(3)^{\frac{1}{2}}=4(1.732)=6.928.$$

The square root of a binomial surd may also be found by inspection if the coefficient of the surd part is 2. (Exercise 81, examples 25–46.)

Ex. 2. Expand $(\sqrt{2}+\sqrt{3})^2$.

By § 154, $$(\sqrt{2}+\sqrt{3})^2=2+2\sqrt{6}+3$$
$$=5+2\sqrt{6}.$$

That is, the square of a binomial surd may be reduced to a binomial surd whose first term is composed of the sum of the first term and last term of the trinomial. The second term is twice the product of the two terms of the binomial which has been squared. (§ 154.)

To extract the square root of a binomial surd: make the coefficient of the irrational term 2, then factor the irrational part so that the sum of the factors gives the rational term.

The square roots of these factors, connected by the sign of the irrational term, is the square root of the binomial surd.

Ex. 3. $\sqrt{5+2\sqrt{6}}=?$

$$\sqrt{6}=\sqrt{2}\cdot\sqrt{3} \text{ and } 3+2=5.$$

Hence, $$\sqrt{5+2\sqrt{5}}=\sqrt{2}+\sqrt{3}.$$

Ex. 4. $\sqrt{14-6\sqrt{5}}$.

$$\sqrt{14-6\sqrt{5}}=\sqrt{14-2\cdot 3\sqrt{5}}=\sqrt{14-2\sqrt{45}},$$

$$\sqrt{45}=\sqrt{9}\cdot\sqrt{5},\ 9+5=14,$$

$$\sqrt{14-2\sqrt{45}}=\sqrt{9}-\sqrt{5}=3-\sqrt{5}.$$

Check: $(3-\sqrt{5})^2=9-6\sqrt{5}+5.$

EXERCISE 84

Find square roots:

1. $5+2\sqrt{15}+3$.
2. $8-2\sqrt{15}$.
3. $8+2\sqrt{15}$.
4. $5-2\sqrt{6}$.
5. $8-\sqrt{60}$.
6. $9+2\sqrt{14}$.
7. $11+2\sqrt{18}$.
8. $11-\sqrt{72}$.
9. $9-\sqrt{80}$.
10. $11+6\sqrt{2}$.
11. $10+4\sqrt{6}$.
12. $27-10\sqrt{2}$.

13. $2+\frac{1}{2}\sqrt{15}=2+2\cdot\frac{1}{4}\sqrt{15}=2+2\sqrt{\frac{1}{16}\cdot 15}=2+2\sqrt{\frac{15}{16}}$

$$=2+2\sqrt{\tfrac{5}{4}\cdot\tfrac{3}{4}}=\tfrac{5}{4}+2\sqrt{\tfrac{5}{4}\cdot\tfrac{3}{4}}+\tfrac{3}{4}.$$

$$\therefore\ \sqrt{2+\tfrac{1}{2}\sqrt{15}}=\sqrt{\tfrac{5}{4}}+\sqrt{\tfrac{3}{4}}=\tfrac{1}{2}\sqrt{5}+\tfrac{1}{2}\sqrt{3}.$$

14. $4+\sqrt{15}$. 15. $1+\frac{2}{5}\sqrt{6}$. 16. $1+\frac{2}{7}\sqrt{10}$. 17. $5+\sqrt{21}$.

18. Find the values in the first five examples correct to three decimal places.

Remember that all numbers have two square roots, alike except in sign.

19. $(a+b)^2-(-a+b)^2=?$ 20. $(a-b)^2-(b-a)^2=?$

21. Is $3-x$ the square root of x^2-6x+9?

22. Is the $\sqrt{2}-\sqrt{3}$ the square root of $5-2\sqrt{6}$? Is $\sqrt{3}-\sqrt{2}$ a square root of this number?

23. $(\sqrt{x-y}+\sqrt{x+y})^2=?$ 24. $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2=?$

Supply the missing term which will make a trinomial square:

25. $x^2 + 14x + ?$

26. $x - 2\sqrt{xy} + ?$

27. $a^{\frac{1}{2}} - 2a^{\frac{1}{4}}b^{\frac{1}{4}} + ?$

28. $5 - 2(15)^{\frac{1}{2}} + ?$

29. $6 + 2(\quad)^{\frac{1}{2}} + 5.$

30. $7 - (\quad)^{\frac{1}{2}} + 3.$

31. $121x^2 - 22x + ?$

32. $(56x)^2 - 2(56x)14 + ?$

33. $(24y)^2 + 2(24y)15 + ?$

34. $(35y)^2 - (70y)12 + ?$

35. $(21x)^2 - (21x)18 + ?$

36. $(13x)^2 - (260x) + ?$

37. $196x^2 - (28x)5 + ?$

38. $225x^2 + (120x) + ?$

39. $49x + 2(7x^{\frac{1}{2}})6 + ?$

40. $36x + 60x^{\frac{1}{2}} + ?$

Division of Irrational Monomials

229. Since $$\sqrt{ab} = \sqrt{a}\sqrt{b},$$

then, $$\frac{\sqrt{ab}}{\sqrt{b}} = \frac{\sqrt{a}\sqrt{b}}{\sqrt{b}} = \sqrt{a}.$$

Ex. 1. Divide $54^{\frac{1}{2}}$ by $27^{\frac{1}{2}}$.

$$54^{\frac{1}{2}} \div 27^{\frac{1}{2}} = (54 \div 27)^{\frac{1}{2}} = 2^{\frac{1}{2}}.$$

Or, $$\frac{54^{\frac{1}{2}}}{27^{\frac{1}{2}}} = \left(\frac{54}{27}\right)^{\frac{1}{2}} = 2^{\frac{1}{2}}.$$

Ex. 2. Divide $9^{\frac{1}{3}}$ by $6^{\frac{1}{2}}$.

$$9^{\frac{1}{3}} = 9^{\frac{2}{6}}\,;\ 6^{\frac{1}{2}} = 6^{\frac{3}{6}}.$$

Then, $$\frac{9^{\frac{1}{3}}}{6^{\frac{1}{2}}} = \frac{9^{\frac{2}{6}}}{6^{\frac{3}{6}}} = \left(\frac{9^2}{6^3}\right)^{\frac{1}{6}} = \left(\frac{3^4}{3^3 \cdot 2^3}\right)^{\frac{1}{6}} = \left(\frac{3}{8}\right)^{\frac{1}{6}}.$$

EXERCISE 85

Divide the following:

1. $\sqrt{108}$ by $\sqrt{54}$.
2. $75^{\frac{1}{2}}$ by $50^{\frac{1}{2}}$.
3. $72^{\frac{1}{3}}$ by $9^{\frac{1}{3}}$.
4. $(243)^{\frac{1}{4}}$ by $3^{\frac{1}{4}}$.
5. $243^{\frac{1}{4}}$ by $3^{\frac{1}{2}}$.
6. $12^{\frac{1}{3}}$ by $12^{\frac{1}{2}}$.
7. $72^{\frac{1}{3}}$ by $72^{\frac{1}{2}}$.
8. $729^{\frac{1}{3}}$ by $729^{\frac{1}{2}}$.
9. $(64\,a^6)^{\frac{1}{5}}$ by $(64\,a^6)^{\frac{1}{10}}$.
10. $24\sqrt[3]{24}$ by $12\sqrt[3]{3}$.
11. $20\sqrt{18}$ by $5\sqrt{6}$.
12. $18\sqrt[4]{243}$ by $3\sqrt[4]{48}$.
13. $\sqrt[3]{\frac{81}{7}}$ by $\sqrt[3]{\frac{27}{49}}$.
14. $\sqrt{\frac{(a+b)^2}{(c+d)}}$ by $\sqrt{\frac{a+b}{(c+d)^3}}$.
15. $\sqrt[3]{\frac{25}{49}}$ by $\sqrt[3]{\frac{75}{98}}$.
16. $\sqrt{\frac{x^2+8\,x+15}{x^2-9}}$ by $\sqrt{\frac{x^2+7\,x+12}{x^2+12\,x-45}}$.
17. $(\frac{14}{18})^{\frac{1}{2}}$ by $2\cdot 7^{\frac{1}{2}}$.
18. $4(\frac{16}{75})^{\frac{1}{3}}$ by $5\cdot(32)^{\frac{1}{3}}$.

Find values:

19. $\sqrt{\sqrt[5]{64}}=(64^{\frac{1}{5}})^{\frac{1}{2}}=(64^{\frac{1}{2}})^{\frac{1}{5}}=8^{\frac{1}{5}}$.
20. $\sqrt[5]{\sqrt{32}}$.
21. $\sqrt[4]{\sqrt[3]{13}}$.
22. $\sqrt{\sqrt[3]{25\,x^2}}$.
23. $\sqrt[3]{27\sqrt{27}}$.
24. $\sqrt[3]{(x+y)^3\sqrt{x+y}}$.
25. $\sqrt[6]{343(a+b)^3}$.
26. $\sqrt[5]{4\sqrt{2}}$.
27. $\sqrt{\sqrt[3]{25\,a^2+90\,ab+81\,b^2}}$.
28. $\sqrt[6]{36\,x^2-12\,x+1}$.
29. $\sqrt[10]{(a+b)^5}$.
30. $\sqrt[7]{128\,a^{14}\sqrt[5]{32\,a^{10}}}$.
31. $\sqrt[5]{2\,a\sqrt[3]{4\,a^2}}$.
32. $\sqrt[3]{27\sqrt{72\,x^2-240\,x+200}}$.
33. $\sqrt[3]{\sqrt[4]{225}}$.

Irrational Equations

230. Sometimes the unknown numbers in an equation are in irrational form. In such a case it is generally preferable to *rationalize the terms* in the equation.

Ex. 1. $(2x-3)^{\frac{1}{3}}-1=2.$

Transposing so that an *irrational alone* composes one side of the equation, we have

$$(2x-3)^{\frac{1}{3}}=3.$$

Cube each side of the equation.

Then, $2x-3=27,$

whence, $2x=30,$

and $x=15.$

Check: $(2\cdot 15-3)^{\frac{1}{3}}-1=2.$

$$(30-3)^{\frac{1}{3}}-1=2.$$
$$(27)^{\frac{1}{3}}-1=2.$$
$$3-1=2.$$

Ex. 2. $(2x+3)^{\frac{1}{2}}+(2x+10)^{\frac{1}{2}}=7.$

Transpose so that one surd (irrational term) composes one side of the equation.

$$(2x+3)^{\frac{1}{2}}=7-(2x+10)^{\frac{1}{2}}.$$

Squaring, $2x+3=49-14(2x+10)^{\frac{1}{2}}+2x+10.$

Transposing, $+14(2x+10)^{\frac{1}{2}}=56$

$$(2x+10)^{\frac{1}{2}}=4.$$

Squaring, $2x+10=16$

$$2x=6$$
$$x=3$$

Check: $(2\cdot 3+3)^{\frac{1}{2}}+(2\cdot 3+10)^{\frac{1}{2}}=9^{\frac{1}{2}}+16^{\frac{1}{2}}=7.$

EXERCISE 86

Solve:

1. $(5x-4)^{\frac{1}{2}}=4.$
2. $(7x-13)^{\frac{1}{3}}=2.$
3. $(7x+6)^{\frac{1}{3}}=3.$
4. $(15x-11)^{\frac{1}{6}}=2.$
5. $(4x+8)^{\frac{1}{2}}=6.$
6. $(9x^2+3x+7)^{\frac{1}{2}}=3x+1.$
7. $(16y^2+12y-11)^{\frac{1}{2}}=4y+1.$
8. $\sqrt{7m-3}+\sqrt{7m}=6.$
9. $(x-4)^{\frac{1}{2}}+x^{\frac{1}{2}}=4.$
10. $(5k+14)^{\frac{1}{2}}+(5k)^{\frac{1}{2}}=7.$
11. $(12x^2-3)^{\frac{1}{2}}=2x\sqrt{3}-3.$
12. $(18x^2-20)^{\frac{1}{2}}=3x\sqrt{2}-2.$

13. $\dfrac{\sqrt{m+9}+\sqrt{m-7}}{\sqrt{m+9}-\sqrt{m-7}}=4.$ (§ 174, 5.)

14. $\dfrac{\sqrt{3x+4}-\sqrt{5x+1}}{\sqrt{3x+4}+\sqrt{5x+1}}=-\dfrac{1}{11}.$

15. $\dfrac{(3x+4)^{\frac{1}{2}}+(3x)^{\frac{1}{2}}}{(3x+4)^{\frac{1}{2}}-(3x)^{\frac{1}{2}}}=\dfrac{5+\sqrt{21}}{5-\sqrt{21}}.$

16. $\dfrac{(3x+4)^{\frac{1}{2}}+(3x)^{\frac{1}{2}}}{(3x+4)^{\frac{1}{2}}-(3x)^{\frac{1}{2}}}=\dfrac{5}{\sqrt{21}}.$

17. $(3v^2+9v-3)^{\frac{1}{2}}-(3v^2+11v-28)^{\frac{1}{2}}=1.$

18. $(27y^3+7)^{\frac{1}{3}}-3y=1.$

CHAPTER XIV

Quadratic Equations

231. An equation of the second degree in one or more unknown numbers is a **quadratic equation.**

A **pure quadratic** contains only the second power of the unknown. Thus, $5x^2 = 125$ is a pure quadratic equation.

An **affected quadratic** contains both the first and second powers of the unknown. Thus, $x^2 - 7x - 12 = 0$ is an affected quadratic equation.

In § 161, we studied the solutions of certain quadratic forms. We shall now study these forms more thoroughly, and extend the work to cover forms which are not readily factored.

REVIEW

1. Write the product of $x + y$ and $x - y$.
2. Translate into English $(x+y)(x-y) = x^2 - y^2$.
3. Translate into English $(5+3)(5-3) = 5^2 - 3^2$.

Write the results of the following:

4. $(2x+7)(2x-7)$.
5. $(3y+4)(3y-4)$.
6. $(7a+4c)(7a-4c)$.
7. $(9x+y)(9x-y)$.
8. $(6m+3n)(6m-3n)$.

9. Check your results by letting $a = 2$, $c = 3$, $m = 13$, $n = 7$, $x = 4$, $y = 7$.

Find prime factors of the following:

10. $64a^2 - 9c^2$.
11. $36x^2 - y^2$.
12. $361 - (18)^2$.
13. $(565)^2 - (564)^2$.
14. $(841)^2 - (839)^2$.
15. Is $2^4 \cdot 6^2 \cdot 7^6 \cdot 3^2$ a square? If so, extract the square root.
16. State rule governing the work of example 1.

Trinomial Squares

17. Write a trinomial square.

18. Is $(x+y)(x+y)$ a square?

19. Define a square.

20. Of what is the square of a binomial composed?

Write the results of the following:

21. $(2x+m)^2$.

22. $(3a+5c)^2$.

23. $(7x+2y)^2$.

24. $(7x-2y)^2$.

25. $(5m+2y)(5m+2y)$.

26. $(15a+17d)(15a+17d)$.

27. $(16x+13y)^2$.

28. $(9a-7c)^2$.

Supply the terms needed to make trinomial squares of the following:

29. x^2+4x.

30. z^2+6z.

31. $4y^2+36$.

32. $16m^2+25c^2$.

33. $8x+16$.

34. $(5x)^2+2(5x)$.

Factor:

35. $(7x)^2+2(7x)6+36$.

36. $(8x)^2+2(8x)9+81$.

37. $(27x)^2+2(27x)14+196$.

38. $(28x)^2-2(28x)23+529$.

Solutions

232. $x^2+4x=21$. What is needed to make the first member a square? Add the required number to each member. Is your equality destroyed? Is the first member a square? The second member? Extract the square root of each member of the equation. Is your equality destroyed? Would it be correct to give your second member a minus sign? A plus sign? Why? How many square roots does a number have? Are they alike? Is the following correct?

$$y^2+6y=7.$$

Adding 9 to each member, $y^2+6y+9=16$.

Extracting the square root, $y+3=+4 \text{ or } -4,$

$$y=-3+4 \text{ or } -3-4$$
$$=1 \text{ or } -7.$$

Substitute these roots in the original equation. Does each root satisfy the equation? How many roots does a first degree equation have? Should a quadratic equation have more than that? How many dimensions has a line? A surface? A solid? Do these answers have any bearing on the number of roots an equation may have?

EXERCISE 87

Solve the following equations, adding what is necessary to make the first member a square: (Check each result.)

1. $x^2+6x=16$.
2. $s^2+4s=32$.
3. $y^2-2y=15$.
4. $z^2-12z=45$.
5. $m^2-8m=-16$.
6. $d^2-d=12$.
7. $y^2-16=0$.
8. $9z^2-25=0$.
9. In the first six equations, how did you know what to add to make the first member a square?

233. If the first degree term is missing, the equation is said to be a **pure quadratic.** A pure quadratic may be solved in either of two ways, *e.g.*

$$(a)\qquad x^2-25=0,$$
$$(x+5)(x-5)=0. \quad (\S\ 161.)$$

If the product of two numbers is zero, one of them must be zero. Since it is unknown which is the zero factor, we place each equal to zero and solve the resulting equations.

Then
$$x+5=0,$$
$$x-5=0.$$
And
$$x=-5,\ x=5.$$

This method is called solution by factoring.

$$(b)\qquad x^2-25=0,$$
$$x^2=25.$$

Extract the square root of each member,

$$x=\pm 5.$$

Hence, $x=5$ or -5 as in solution (a).

We take for our type $a^2 + 2ab + ?$

Where the first term is a square.

The trinomial square is the square of a binomial.

The first term is the square of the first term of the binomial, the third term is the square of the second term of the binomial, the middle term is twice the product of the first and second terms of the binomial. Therefore, the square root of the first term (of the trinomial) multiplied by 2 and divided into the middle term of the trinomial should give the second term of the binomial. This quotient is what we must square and add in order to complete the square. Try this on $a^2 + 2ab$: the square root is a, twice this root is $2a$, $2ab \div 2a = b$, the square of which gives the proper term to be added.

EXERCISE 88

Use this method on the following:

1. $x^2 - 8x$.
2. $m^2 - 2m$.
3. $9x^2 + 6x$.
4. $64x^2 + 48x$.
5. $25m^2 - 25m$.
6. $16z^2 + 32z$.
7. $9x^2 + 12x = 32$.
8. $36x^2 + 4x = 7$.

Find two roots for each equation by substituting values for x between -4 and $+6$:

9. $3x^2 - 9x - 11 = 2x^2 - 4x - 15$.
10. $8x^2 + 5x - 14 = 7x^2 + 6x - 8$.
11. $-2x^2 + 12x - 18 = 8x - 21 - 3x^2$.
12. $7x^2 + 22x - 18 = 6x^2 + 26x - 22$.
13. $5x^2 - 9x + 13 = 3x^2 - 21x - 5$.

Composition of Squares

234. Is the product of two squares always a square?

Is $a^n b^m c^k$ a square? What values must n, m, k have if it is a square?

(*a*) If a number is multiplied by four times itself, is the result a square?

(*b*) If a number is multiplied by nine times itself, is the result a square?

(*c*) If a number is multiplied by twenty-five times itself, is the result a square?

Try these questions on 1, 2, 3, 4, 5, 6, 7. Are they squares which you recognize? If you multiply a by ab^2, is the result a square? Why? It will be seen, then, that if we can make the first term of a quadratic expression a square, the method used in exercise 88 may be followed.

Ex. 1. $3x^2 + 7x = 6$.

The first term becomes a square by multiplying the equation by $1 \cdot 3$, $4 \cdot 3$, $9 \cdot 3$, $25 \cdot 3$, etc. However, if we use another multiplier than $4 \cdot 3$, the quotient b obtained by the method used in exercise 88 may be a fraction. Since we obtain this quotient by dividing by twice the square root of the coefficient of x^2, the Hindoos learned that if 4 was the square factor used in the process, the quotient to be squared was always an integer.

Then to make the first term a square we multiply each term by $3 \cdot 4$. Since the second term always contains the factor 2 (twice the product), it is simpler to multiply the first term by $3 \cdot 4$ and the second term by $2 \cdot 6$.

$$\begin{array}{l} 3x^2 + 7x = 6 \\ \underline{3 \cdot 4 \qquad\qquad} \\ \underline{3^2 \cdot 2^2 \cdot x^2 \quad} \end{array}$$

The square root of our first term is now $3 \cdot 2 \cdot x$ or $6x$. *$6x$ is the unknown*, that is, the a of $a^2 + 2ab$. Write the first term in the form $(6x)^2$. Multiply the second term by $2 \cdot 6$. We have $2 \cdot 6 \cdot 7x$ or $2(6x)7$ for the second term, and $6 \cdot 3 \cdot 4$ for the absolute term.

$3x^2 + 7x = 6$ is now transformed into an equivalent equation: $(6x)^2 + 2(6x)7 = 6 \cdot 3 \cdot 4$.

It is evident that b in this equation is 7, and that 49 or 7^2 must be added to both sides of the equation.

$$(6x)^2 + 2(6x)7 + 49 = 72 + 49 = 121.$$
$$6x + 7 = \pm 11$$
$$6x = 4 \text{ or } -18$$
$$x = \tfrac{2}{3} \text{ or } -3.$$

Ex. 2. $24x^2 - 38x = -15$.

Multiply by $24 \cdot 4$,

$$\begin{array}{l} 24x^2 - 38x = -15. \\ \underline{24 \cdot 4} \\ \underline{24^2 \cdot 2^2 \cdot x^2} \\ (48x)^2 - 2(48x)38 = -15 \cdot 24 \cdot 4 \\ (48x)^2 - 2(48x)38 + 38^2 = -1440 + 38^2 = 4 \end{array}$$

$$\begin{aligned} 48x - 38 &= \pm 2 \\ 48x &= 38 \pm 2 \\ x &= \tfrac{40}{48} \text{ or } \tfrac{36}{48} \\ &= \tfrac{5}{6} \text{ or } \tfrac{3}{4}. \end{aligned}$$

The advantage of indicating the multiplication in this manner is that no matter how large the number involved, no multiplication is ever necessary except in the second member, and factoring may sometimes avoid it there.

Ex. 3. $15x^2 - 8x = 63$.

$$\begin{array}{l} \underline{15 \cdot 4} \\ \underline{15^2 \cdot 2^2 \cdot x^2} \\ (30x)^2 - 2(30x)8 = 63 \cdot 15 \cdot 4 \\ (30x)^2 - 2(30x)8 + 8^2 = 63 \cdot 15 \cdot 4 + 8^2 \end{array}$$

$$\begin{aligned} &= 63 \cdot 15 \cdot 4 + 4^2 \cdot 2^2 \\ &= 4(63 \cdot 15 + 16) \\ &= 4(961)^* \end{aligned}$$

$$\begin{aligned} 30x - 8 &= \pm 2 \cdot 31 \\ 30x &= 70 \text{ or } -54 \\ x &= \tfrac{7}{3} \text{ or } -\tfrac{9}{5}. \end{aligned}$$

EXERCISE 89

Solve the following equations in this manner:

1. $3x^2 - 19x = 14$.
2. $10x^2 - 29x = 21$.
3. $6x^2 + 13x = -6$.
4. $9x^2 - 21x = 18$.
5. $4x^2 - 4x = 63$.
6. $x^2 + 18x = 56$.
7. $x^2 - 2x = 99$.
8. $25x^2 + 80x = -64$.
9. $28x^2 + 32x = 48$.
10. $22x^2 - 25x = 12$.

* Avoid multiplication if possible.

11. $2y^2 - 5y + 4 = 0.$

12. $ax^2 + bx + c = 0.$

13. $3x^2 = 15 - 4x.$

14. $\frac{x+1}{x-1} - \frac{x-2}{x+2} = \frac{9}{5}.$

15. $\frac{3}{2(x-1)} - \frac{1}{4(x+1)} = \frac{1}{8}.$

16. $\frac{x+1}{x+2} + \frac{x-1}{x-2} = \frac{2x-1}{x-1}.$

17. $\frac{x-1}{x+1} + \frac{x-2}{x+2} = \frac{2x+13}{x+16}.$

18. $4ax = (a^2 - b^2 + x^2).$

19. $\frac{x}{a} + \frac{a}{x} = \frac{x}{b} + \frac{b}{x}.$

20. One number is 7 greater than another and the sum of their squares is 337. Find the numbers.

21. A certain number added to its square is 72. Find the number.

22. One number is three greater than another and the difference between their cubes is 387. Find the numbers.

23. A piece of tin is in the form of a square 24″ on a side. A small square is cut out from each corner of the sheet and the sides of the sheet folded up to form an open box whose volume is equivalent to four times the cube of the side of the square cut out increased by 768 cubic inches. Find the dimensions of the box. (Draw to a scale.)

24. Three times a certain number added to twice its square is 72. Find the number.

25. One side of a rectangle is 21′ greater than the other. The diagonal is 39′. Find the area.

26. One side of a rectangle is 9′ more than the other, the area is 360 square feet. Find the dimensions.

27. The length of a rectangular piece of ground is four times its width. A walk 6′ wide surrounds this rectangle; the area of the rectangle exceeds that of the walk by 72 square feet. Find the dimensions of the piece of ground.

28. A line PD drawn from P, a point in the circumference of a circle perpendicular to the diameter of the circle, is a mean proportional between the segments, AD and DB, of the diameter. If PD is 32′ and AB is 80′, find AD and DB. Why is there more than one solution?

29. $\dfrac{6x^2-7x-20}{21x^3+12x^2-7x+9}=0.$ Solve for x.

30. Two men, A and B, leave a point C at the same time. A travels north 7 miles an hour faster than B travels east. At the end of three hours they are 39 miles apart. Find the rate of each.

31. The area of an equilateral triangle is $25\sqrt{3}$. Find one side.

32. It has been found that when a body falls from rest, the distance passed over is 16.075 times the square of the time in seconds. That is, if S = distance and t = seconds, the equation for the above law is

$$S=16.075\,t^2.$$

How far will a body fall in three seconds?

33. The Washington monument is 555′ high. A juggler claimed that he could stand on the ground, hold a knife in his teeth, and impale upon the knife an apple dropped from the top of the monument. From the time when the apple was dropped how many seconds in which to locate it did he have before it reached his knife?

34. The Masonic Temple at Chicago is 384′ high. If a stone is dropped from the roof, what time does it take to reach the pavement?

35. An airship is sailing one fourth mile above the earth. An anchor is lost overboard. How long does it take the anchor to reach the ground?

36. Solve: $25(a+b)^2-40(a+b)+16=0.$

37. Solve: $(2x+3)^2+18(2x+3)=40$.

38. Solve: $\sqrt{3x+8}=\sqrt{5x-7}$.

39. Solve: $(9x^2+15)^{\frac{1}{2}}=(4x^2+140)^{\frac{1}{2}}$.

40. Solve: $\sqrt{\dfrac{3x+2}{3x-2}}=\dfrac{\sqrt{5x+2}}{\sqrt{5x-7}}$.

41. Solve: $\sqrt{\dfrac{x+5}{x-5}}-\sqrt{\dfrac{x-5}{x+5}}=0$.

42. The diameter *AB* of a circle is 117′. The perpendicular *PD* drawn from a point *P* in the circumference to *AB* is 54′. Find *AD* and *DB*.

43. The radius of a circle is $84\frac{1}{2}$′. At a point *D* in the diameter, 25′ from one end of the diameter, a perpendicular *DP* is drawn intersecting the circumference at *P*. Find *DP*.

44. One side of a right triangle is 8′ more than the other. The hypotenuse is 40′. Find the area.

45. A triangle is formed by the *X*-axis and the lines $2x+y=20$, $2x-y=-20$. A rectangle is inscribed in this triangle having a base of 10, coincident with the *X*-axis. Find the area of the rectangle.

46. Show that $\dfrac{\dfrac{\pi R^2}{2}-\dfrac{\pi}{2}\left(\dfrac{R-x}{2}\right)^2+\dfrac{\pi}{2}\left(\dfrac{R+x}{2}\right)^2}{\dfrac{\pi R^2}{2}-\dfrac{\pi}{2}\left(\dfrac{R+x}{2}\right)^2+\dfrac{\pi}{2}\left(\dfrac{R-x}{2}\right)^2}=\dfrac{R+x}{R-x}$.

47. Solve: $\dfrac{(m+3)^{\frac{1}{2}}}{m^{\frac{1}{2}}}+\dfrac{m^{\frac{1}{2}}}{(m+3)^{\frac{1}{2}}}=\dfrac{5\sqrt{6}}{6}$.

48. $\sqrt[3]{5x^2-14x+8}=6$. Solve for x.

49. The bases of a trapezoid are $2x+7$ and $3x+6$, respectively. The altitude is $x+2$ and the area is 99. Find the dimensions. Is there more than one trapezoid? Why?

50. In a trapezoid $ABCD$, DE is the altitude. AB and DC are the parallel sides. AB is 12 more than twice DE, and DC is 16 less than 12 times DE. The area is 2324. Find the dimensions.

51. I have two consecutive numbers, the sum of whose squares is 1257. Find the numbers.

52. Find two consecutive even numbers the difference between whose cubes is 488.

53. The difference between two numbers is .3 and the difference between their cubes is .117. Find the numbers.

Graphs

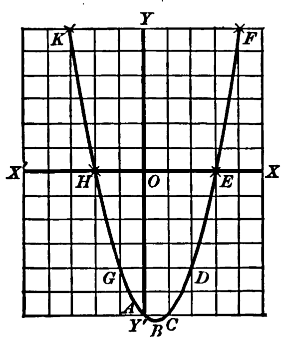
235. We have found that every number has two square roots, also that a quadratic equation has two roots. The graph gives a geometric picture showing the reason for this.

In § 112 we noticed that an equation of the first degree represented a straight line and that therefore two first degree equations had one solution. (The straight lines could intersect in but one point.)

Ex. 1. Plot $x^2 - x - 6 = 0$.

This equation contains but one unknown because $y = 0$. Substituting y for 0, we have $y = x^2 - x - 6$.

Substituting values for x, we have the values



Scale $\frac{1}{4}'' = 1$ unit.

x	y	
0	-6	(A)
$\frac{1}{2}$	$-6\frac{1}{4}$	(B)
1	-6	(C)
2	-4	(D)
3	0	(E)
4	6	(F)
-1	-4	(G)
-2	0	(H)
-3	6	(K)

We see that an equation of the second degree is a curve. This is true also of any equation of higher degree than the second.

Suppose we solve $x^2 - x - 6 = 0$.

$$(x-3)(x+2) = 0,$$
$$x = 3 \text{ or } -2.$$

Now note points H and E. H is at $x = -2$, E is at $x = 3$. These points where the curve crosses the X-axis correspond to the roots of the equation. This is in general true, although sometimes the crossing points are imaginary.

In a quadratic equation the roots may be:

(*a*) Real and unequal.

(*b*) Real and equal.

(*c*) Imaginary.

In the example just solved, the curve crosses the axis in two distinct points; that is, the roots are *real* and *unequal* (*a*).

Ex. 2. Plot $x^2 - 2x + 1 = 0$.

Place $y = 0 = x^2 - 2x + 1$.

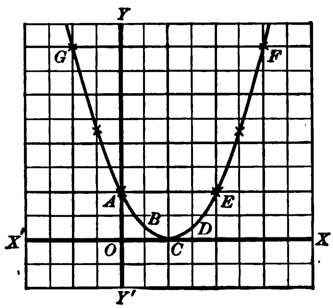
Note that no values of x will make y negative. That is, the curve does not extend below the X-axis.

x	y	
0	1	(A)
$\frac{1}{2}$	$\frac{1}{4}$	(B)
1	0	(C)
$\frac{3}{2}$	$\frac{1}{4}$	(D)
2	1	(E)
3	4	(F)
-1	4	(G)

Solving, $x^2 - 2x + 1 = 0$,

$$(x-1)(x-1),$$
$$x = 1 \text{ or } 1.$$

The roots are *real* and *equal* (*b*).

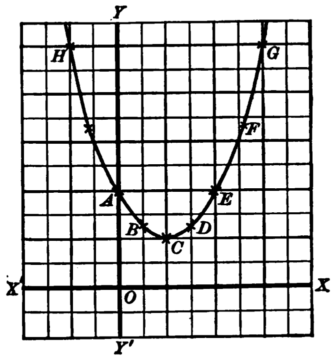


Scale $\frac{1}{4}'' = 1$ unit.

In this example the two points where the curve crosses the X-axis have come together. The X-axis is said to be tangent

to the curve. If you should raise the curve in example 1 (move it in a positive direction upwards), the points H and E would approach each other until they would fall together at $x = \frac{1}{2}$. Should you raise the curve entirely above the X-axis, the intersection would be imaginary (would not exist) and the roots would be imaginary. Let us examine a case of this kind.

Ex. 3. Plot $x^2 - 2x + 2 = 0$.



x	y	
0	2	(A)
$\frac{1}{2}$	$1\frac{1}{4}$	(B)
1	1	(C)
$\frac{3}{2}$	$1\frac{1}{4}$	(D)
2	2	(E)
$\frac{5}{2}$	$3\frac{1}{2}$	(F)
3	5	(G)
-1	5	(H)

Here the curve does not cross the X-axis. The intersection of the curve and the X-axis is *imaginary* (c).

Solving,
$$x^2 - 2x + 2 = 0,$$
$$x^2 - 2x = -2,$$
$$x^2 - 2x + 1 = -1,$$
$$x - 1 = \pm\sqrt{-1},$$
$$x = 1 \pm \sqrt{-1}.$$

This gives us a new kind of number, the square root of a negative number. We have found that the squares of all real numbers are positive, but the square of this number is negative. We have, therefore, found a new unit, and it is called an *imaginary number.* Any even root of a negative number is imaginary.

This new unit is such that its square is negative.

E.g. $(\sqrt{-1})^2 = [(-1)^{\frac{1}{2}}]^2 = (-1)^{\frac{2}{2}} = -1.$

$(\sqrt{-2})^2 = (\sqrt{2}\sqrt{-1})^2 = [2^{\frac{1}{2}} \cdot (-1)^{\frac{1}{2}}]^2 = 2(-1) = -2.$

These imaginary numbers often occur in the solution of quadratic equations.

EXERCISE 90

Plot the graphs of the following equations.

In each case solve the equation algebraically and compare solutions with results in your graph.

1. $x^2 - 6x + 8 = 0.$
2. $x^2 + x - 6 = 0.$
3. $x^2 + 2x + 2 = 0.$
4. $x^2 - 4 = 0.$
5. $x^2 - 9 = 0.$
6. $x^2 + 9 = 0.$
7. $4x^2 - 9 = 0.$
8. $x^2 - 6x + 9 = 0.$
9. $x^2 + 4x + 4 = 0.$
10. $x^2 - x + 1 = 0.$

236. It is possible to tell the *nature of the roots* of a quadratic without solving the equation.

When we solved example 12, exercise 89, we found,

$$(1)\quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \text{ and}$$

$$(2)\quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The equation from which these roots were obtained is

$$ax^2 + bx + c = 0,$$

where a is the coefficient of x^2,

b is the coefficient of x,

c is the absolute term in the first member, and the second member is zero.

In roots (1) and (2) the expression $\sqrt{b^2 - 4ac}$ occurs, the difference in these roots being that in root (1) the radical is added, in root (2) it is subtracted.

(*a*) If b^2-4ac is positive, that is, $b^2-4ac>0$, then $b^2>4ac$, and there is a real number to add to the $-b$ in the numerator. The roots are then *real and unequal.*

(*b*) If $b^2-4ac=0$, that is, if $b^2=4ac$, there is nothing to add to the $-b$, the roots are each $-\dfrac{b}{2a}$; namely, the roots are *real and equal.*

(*c*) If $b^2-4ac<0$, that is, if $b^2<4ac$, the quantity under the radical sign is negative, and the roots *imaginary.* Imaginary roots always go in pairs.

Try this test on:

Example 2, exercise 90, $a=1$, $b=1$, $c=-6$,
$b^2-4ac=1+24=+$, (real and unequal).

Example 9, exercise 90, $a=1$, $b=4$, $c=4$,
$b^2-4ac=16-16=0$, (equal).

Example 10, exercise 90, $a=1$, $b=-1$, $c=+1$,
$b^2-4ac=1-4=-$, (imaginary).

EXERCISE 91

Find the nature of the roots of the following:

1. $4x^2+4x+1=0$.
2. $4x^2+8x+1=0$.
3. $4x^2-12x+9=0$.
4. $4x^2-12x+12=0$.
5. $2x^2-8x=9$.
6. $7x^2-4x-8=0$.
7. $7x^2+4x+8=0$.
8. $3x^2+5x-6=0$.
9. $3x^2+14=16$.
10. $5x^2-7x+14=0$.

11. Solve examples 1 to 10, using the roots of $ax^2+bx+c=0$ (§ 236) as a formula.

E.g. Solve $5x^2-7x=2$.

Reducing to the form $ax^2+bx+c=0$,

$5x^2-7x-2=0$. Here $a=5$, $b=-7$, $c=-2$.

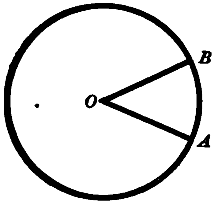
$$x=\frac{7+\sqrt{49-4\cdot 5(-2)}}{10}=\frac{7+\sqrt{89}}{10},$$

$$x=\frac{7-\sqrt{49-4\cdot 5(-2)}}{10}=\frac{7-\sqrt{89}}{10}.$$

CHAPTER XV

The Circle

237. We learned in § 62 that the circle is a plane figure bounded by a curve all points of which are equally distant from a point within called the center.



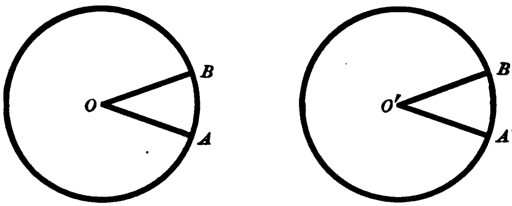
A **central angle** is an angle whose vertex is at the center of a circle and whose sides are radii of the circle.

We will assume that the number of degrees in the angle is equal to the number of degrees in the arc. That is, number of degrees in $\angle O =$ number of degrees in arc AB.

A **quadrant** is one fourth of a circumference.

238. We can now state the proposition, that *in the same circle or in equal circles, equal central angles intercept equal arcs.*

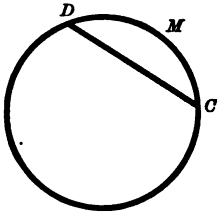
In the equal circles whose centers are at O and O', respectively, show by superposition that $AB = A'B'$.



The converse of this proposition is also true.

239. It follows from § 238 that in the same circle or in equal circles the *greater central* angle intercepts the greater arc.

240. Show that *a diameter bisects the circle and the circumference.*

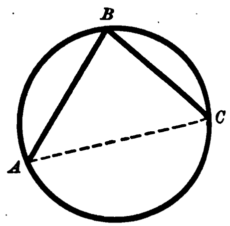


241. A **chord** is a straight line joining any two points of a circumference, as CD.

A **segment** is the portion of a circle between a chord and its arc, as CDM.

The arc CMD is *subtended* by the chord CD. Every chord subtends two arcs. Illustrate.

242. *A straight line cannot meet a circumference in more than two points.*



243. An **inscribed angle** is an angle whose vertex is on the circumference and whose sides are chords; as $\angle ABC$. The angle ABC is also said to be inscribed in the segment ABC. The angle A is inscribed in segment CAB.

THEOREM LVIII

244. *In the same circle or in equal circles, equal arcs are subtended by equal chords.*

Use figure in § 238. Draw chords BA and $B'A'$. Prove

$$\triangle OAB = \triangle O'A'B'.$$

245. The converse of this theorem is also true. State the converse.

THEOREM LIX

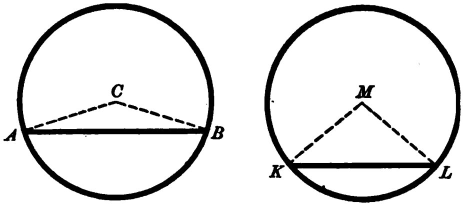
246. *In the same circle or in equal circles, the greater of two arcs is subtended by the greater chord, each being less than a semi-circumference.*

In the equal circles whose centers are C and M, arc AB $>$ arc KL.

To prove chord AB $>$ chord KL.

Draw radii AC, BC, KM, LM.

Prove by applying Theorem XXII to $\triangle$s ABC and KLM.



247. The converse of this theorem is true. State the converse.

THEOREM LX

248. *A diameter perpendicular to a chord bisects the chord and its subtended arcs.*

Draw $\odot$ with center O. Draw chord AB and diameter $CD \perp AB$ intersecting AB at E. We have

Given $\odot$ DBA with $DC \perp AB$.

To prove $AE = EB$, arc $AD =$ arc DB, and arc $AC =$ arc CB.

Proof. 1. Draw OA and OB.

2. Prove $\triangle AEO = \triangle EBO$.

3. $\therefore AE = EB$.

4. Apply § 238 to show that two of the arcs are equal.

THEOREM LXI

249. *In the same circle or in equal circles, equal chords are equally distant from the centers.*

Draw two equal circles with centers at O and O'. We have

Given equal circles with centers at O and O', and chord $AB =$ chord CD.

To prove AB and CD equally distant from O and O', respectively.

Proof. 1. Draw $OK \perp AB$ and $O'M \perp CD$. Draw OA and $O'C$.

2. Prove $\triangle OKB = \triangle O'MC$.

EXERCISE 92

1. A, B, C, D, are points in succession on a circumference such that chord $AB =$ chord CD. Show that

$$\text{chord } AC = \text{chord } BD.$$

2. A central angle AOB is bisected by a radius OC. Let A', C', B', be the middle points of OA, OC, OB, respectively. Prove that broken line $A'C'B' =$ chord AC.

3. Draw two equal intersecting chords. Connect the center of the circle with their middle points and their point of intersection. Prove that two equal triangles are formed.

4. Draw a line through the middle points of a chord and one of its arcs. Show that this line passes through the center of the circle.

5. Prove that in the same circle or equal circles chords equally distant from the center are equal.

THEOREM LXII

250. *If two chords are unequal, the greater chord is nearer the center.*

Draw $\odot ABC$ with center at O.

Draw chord $AB >$ chord AC. We have

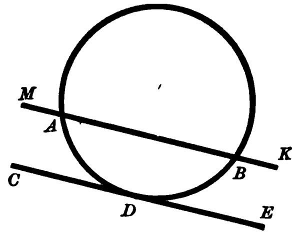
Given $\odot ABC$ with chord $AB >$ chord AC.

To prove chord AB nearer O than chord AC.

Proof. 1. Draw OM and OK respectively perpendicular to AB and AC. Draw MK. Represent $\angle MKO$ by x; $\angle KMO$, y; $\angle AKM$, o; $\angle AMK$, z.

2. $AM > AK$. ($AB > AC$, § 248.)
3. Then $o > z$. (§ 129.)
4. But $o + x = 90°$. (?)
5. And $z + y = 90°$.
6. Then $y > x$.
7. *And* $OK > OM$.

251. A **secant** is a line intersecting a circumference in. two points, as MK.



A **tangent** is a line which can touch a circumference in one point only, as CE.

One can see that if line MK is moved toward D, the points A and B approach each other until they become two consecutive points D. A tangent is thus the limiting position of a secant.

THEOREM LXIII

252. *A line perpendicular to a radius at its extremity is tangent to the circle.*

Draw circle with center O.

Draw radius OA. Through A draw $CD \perp OA$. We have

Given circle with center O, $CD \perp$ radius OA at A.

To prove CD tangent to the circle.

Proof. 1. Draw OH, any other line meeting CD at H.

2. Then $OH > OA$. (§ 129.)

3. Hence H lies without the circumference because OH is greater than a radius, and since H is any point in CD except A, CD is a tangent.

253. As a consequence of §§ 250 and 252,

I. *Of two chords unequally distant from the center, the one nearer the center is the greater.*

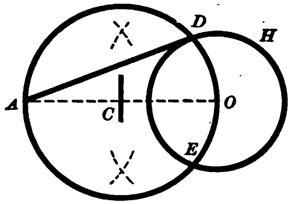
II. *A tangent to a circle is perpendicular to a radius at the point of contact.*

III. *A line perpendicular to a tangent at the point of contact passes through the center of the circle.*

Problem

254. *To draw a tangent to a circle from an external point.*

Given $\odot EHD$ with center O, and A any external point.



To draw a tangent from A to the circle.

Construction. 1. Draw AO.

2. Find C, middle point of AO.

3. With C as center and CO as radius, describe a circumference cutting $\odot EHD$ at E and D.

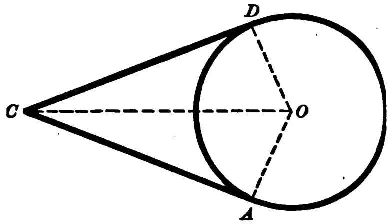
4. E and D are points of tangency.

5. AD is the required tangent.

Theorem LXIV

255. *Two tangents to a circle from an outside point are equal.*

Draw circle with center O. Draw tangents CD and CA. We have,



Given tangents CD and CA drawn from C to a circle with center O.

To prove $CD = CA$.

Proof. Draw CO, DO, AO.

(The proof is left to the pupil.)

Theorem LXV

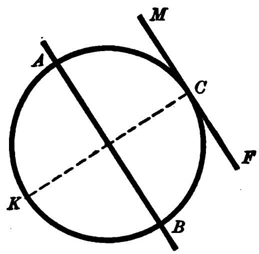
256. *Two parallels intercept equal arcs on a circumference.*

These parallels may have three different positions with reference to the circle,

I. *When one line is a tangent and the other a secant.*

II. *When both lines are secants.*

III. When both lines are tangents.



I. **Given** tangent MF and secant AB.

To prove arc $BC =$ arc CA.

Proof. 1. Draw diameter CK.

2. Then $AC = CB$. (§ 248.)

II. Draw secant $DE \parallel AB$ between AB and MF.

Show that $DC = CE$. Subtract from $AC = CB$.

III. Draw tangent GH through K. Then use MF, AB, GH.

EXERCISE 93

1. The angle between two tangents drawn from a common external point is bisected by the line joining this point to the center of the circle.

2. The angle between two tangents drawn from a common point is the supplement of the angle formed by radii drawn to the points of contact.

3. If two circles are tangent externally and a common tangent is drawn through their point of contact, tangents drawn to each circle from any point in their common tangent are equal.

4. If two unequal circles are tangent externally and a common tangent is drawn through their point of contact, and any line is drawn across the common tangent and tangent to each circle, the distance between the points of contact of this line is bisected by the common tangent.

5. If any two circles are tangent either externally or internally, the line joining their centers passes through the point of tangency. (§ 252.)

6. Two tangents to a circle meet $24\sqrt{2}''$ from center of circle, and form an angle of 90°. Find length of tangent.

Theorem LXVI

257. *Three points not in the same straight line determine a circumference.*

(Remember we can impose only two conditions on a straight line. A line can be drawn through two points; through one point and in a given direction.)

Given points A, B, C not in the same straight line.

To prove only one circumference can be drawn through these three points.

Proof. 1. A, B, C, may be considered the vertices of a triangle.

2. The perpendicular bisectors of the sides of this triangle meet in a point equidistant from the points A, B, C. (§ 143.)

3. This point of intersection is therefore the center of a circumference passing through A, B, C.

4. Again, the center of the circumference must be in each of the perpendicular bisectors of two of the sides AB, AC, or BC; and as they can intersect in but one point, only one circumference can be drawn through A, B, C. (§ 248.)

Note that in the *circle theorems*, you are using all of the *straight line theorems.*

Theorem LXVII

258. *If two circumferences intersect, the straight line joining their centers bisects their common chord at right angles.*

Draw Ⓢ with centers C and C'. Draw the common chord AB. We have,

Given circles with centers C and C' intersecting at A and B, and common chord AB.

To prove $CC' \perp AB$ at middle point.

Proof. (*Prove* CC' bisects AB at right angles.) (§ 95, II.)

259. An **inscribed polygon** is a polygon each of whose vertices lies in a circumference. The circle is circumscribed about the polygon.

260. A **circumscribed polygon** is a polygon each of whose sides is tangent to the circle.

EXERCISE 94

1. If two circles intersect, the line joining their centers is greater than the difference between their radii.

2. If at the extremity of any chord, not a diameter, lines are drawn each perpendicular to the chord, equal arcs are intercepted between the perpendiculars.

3. A trapezoid inscribed in a circle is isosceles. Is the center of the circle within or without the trapezoid?

4. Join the middle points of two unequal parallel chords. Show that this line passes through the center of the circle.

5. Two equal parallel chords form the opposite sides of a rectangle or a square. Can the center of the circle lie outside this quadrilateral?

6. A rectangle is inscribed in a circle whose radius is 41″. One side of the rectangle is 9″ from the center of the circle. Find the area of the rectangle.

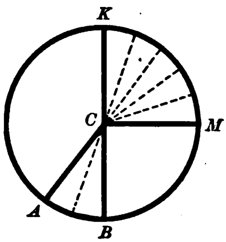
7. A rectangle is inscribed in a circle. One side of the rectangle is 32′; its area is 1920 square feet. Find the radius of the circle and the distance of the sides of the rectangle from the center.

8. The area of a rectangle inscribed in a circle is $x^2 - 2x - 48$ square inches. The longer side of the rectangle is 5″ from the center of the circle. Find the dimensions of the rectangle, and the radius of the circle.

THEOREM LXVIII

261. *In the same circle or in equal circles, central angles are in the same ratio as their intercepted arcs.*

Draw a circle with center at C. Draw $\angle s\ ACB$ and MCK. We have



Given circle $ABMK$ and central angles ACB and MCK.

To prove $\dfrac{\angle ACB}{\angle MCK}=\dfrac{\text{arc } AB}{\text{arc } MK}$.

Proof. 1. Find a common measure of arcs AB and MK.

2. Apply this measure to arcs AB and MK. Suppose this measure is contained twice in AB and five times in MK.

3. Draw radii to the points of division of these arcs.

4. Then $\dfrac{\angle ACB}{\angle MCK}=\dfrac{2}{5}$.

5. And $\dfrac{\text{arc } AB}{\text{arc } MK}=\dfrac{2}{5}$.

6. $\therefore$ $\dfrac{\angle ACB}{\angle MCK}=\dfrac{\text{arc } AB}{\text{arc } MK}$. (?)

This method of proof holds for any number of subdivisions of arcs AB and MK.

This theorem is another way of stating that a central angle is measured by the arc it intercepts. (§ 237.)

EXERCISE 95

1. Two equal circles intersect in such manner that the radius of each circle drawn to the point of intersection of the circles is a tangent to the other circle. The length of the common chord is 16″. Find the radius of the circle, and the distance between the centers.

2. Two unequal circles intersect in such manner that the radii drawn to the point of intersection make angles of 60° and 30° with the line of centers. Distance between centers is 16″. Find the radius of each circle.

3. Two tangents to a circle intersect at an angle of 120°. The length of one tangent is 18″. Find the length of the chord of contact (between points of contact) and the radius of the circle.

4. Two tangents intersect at an angle of 90°. The length of one tangent is 18″. Find the radius of the circle and the chord of contact.

5. Two tangents intersect at an angle of 90°, the length of one tangent is $x + a$. Find the radius of circle and chord of contact. Check your results making $x = 12''$, $a = 6''$.

6. Through center O, draw diameter AB. Draw chord BC and radius OC. Show that angle AOC is equal to twice $\angle B$. If arc AC is equal to 90°, find $\angle B$.

7. A rectangle is inscribed in a circle whose radius is 17″. One side of the rectangle is 30″. Find the other dimension of the rectangle.

8. A line drawn from the center of a circle perpendicular to a chord is 16″ long. The chord is 60″ long. Find the radius of the circle.

9. An arc is one sixth of a circumference. The chord joining the extremities of the arc is 18″ long. How far is the chord from the center of the circle?

10. A chord 12″ long is 8″ from the center of a circle. Find the number of degrees in the arc which the chord subtends.

11. An isosceles trapezoid is inscribed in a circle whose radius is 17″. The longer base of the trapezoid is 30″, and the shorter base is 4″ farther from the center of the circle than the longer base. Find the shorter base.

Theorem LXIX

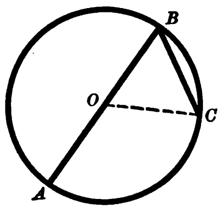
262. *An inscribed angle is measured by one half its intercepted arc.* (The center of the circle may have three positions with respect to the sides of the angle.)

I. *When one side of the angle passes through the center.*

II. *When the center lies within the angle.*

III. *When the center lies without the angle.*

I. Draw inscribed angle ABC, AB being a diameter. We have



Given inscribed angle ABC.

To prove $\angle ABC$ is measured by $\frac{1}{2}$ arc AC. (AC being measured in degrees.)

Proof. 1. Draw OC.

2. What kind of a triangle is OCB?

3. $\angle AOC = \angle C + \angle B$. (?)

4. $\angle AOC$ is measured by arc AC. (?)

5. Then $\angle B$ is measured by $\frac{1}{2}$ arc AC. (?)

II. Draw inscribed angle ABC, center O lying within the angle. We have

Given inscribed $\angle ABC$.

To prove $\angle ABC$ is measured by $\frac{1}{2}$ arc AC.

Proof. 1. Draw diameter BK.

2. Find measure of $\angle ABK$.

3. Find measure of $\angle KBC$.

4. Add equations formed in 2 and 3.

III. Draw inscribed angle ABC, center O lying without the angle. Draw diameter BK. Find measure of $\angle$s KBC and KBA.

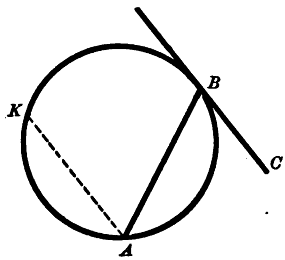
(The proof is left *to* the pupil.)

THEOREM LXX

263. *The angle formed by a tangent and a chord is measured by one half the intercepted arc.*

Given $\angle ABC$ formed by chord AB and tangent BC.

To prove $\angle ABC$ is measured by $\frac{1}{2}$ arc AB.



(In this type of theorem measurement of arc AB is considered as the number of degrees in the arc.)

Proof. 1. Draw chord $AK \parallel BC$.

2. Then $\angle A = \angle ABC$. (?)

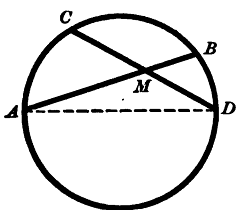
3. $\angle A$ is measured by $\frac{1}{2}$ arc BK. (§ 262.)

4. But arc $BK =$ arc AB. (?)

5. Then $\angle ABC$ is measured by $\frac{1}{2}$ arc AB. (?)

THEOREM LXXI

264. *The angle formed by two intersecting chords is measured by one half the sum of the intercepted arcs.*



Given chords AB and CD intersecting at M within the circle.

To prove AMC is measured by one half the sum of the arcs CA and DB.

Proof. 1. Draw chord AD.

2. $\angle AMC = \angle A + \angle D$. (?)

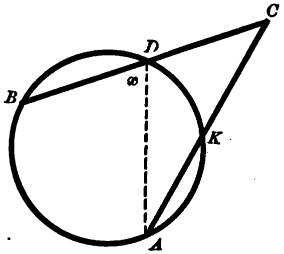
3. But $\angle A$ is measured by $\frac{1}{2}$ DB and $\angle D$ is measured by $\frac{1}{2}$ CA. (?)

4. Then $\angle AMC$ is measured by $\frac{1}{2}(CA + DB)$.

THEOREM LXXII

265. *The angle formed by two secants is measured by one half the difference between the intercepted arcs.*

Draw secants CB and CA intersecting the circumference at D and K, respectively. We have



Given $\angle C$ formed by secants CB and CA intercepting arcs BA and KD.

To prove $\angle C$ is measured by $\frac{1}{2}(BA - KD)$.

Proof. 1. Draw DA. Call $\angle ADB, x$.

2. $\angle x = \angle C + \angle A$ or $C = x - A$. (?)

3. But x is measured by $\frac{1}{2}BA$ and A is measured by $\frac{1}{2}KD$. (?)

4. Then $\angle C$ is measured by $\frac{1}{2}(BA - KD)$.

EXERCISE 96

1. An inscribed angle intercepts 48° 30′ on the circumference. How large is the angle?

2. Two chords intersecting intercept arcs of 34° and 47°, respectively. What angle is formed by the chords?

3. Two secants form an angle of 28°. One of the arcs intercepted is 49°. What is the other intercepted arc? Is there more than one answer to this question? Draw diagram showing why.

4. In what segment may a right angle be inscribed?

5. The angle between a tangent and a chord is equal to the angle inscribed in the opposite segment.

6. The angle formed by a tangent and secant is measured by one half the difference between the intercepted arcs.

7. The angle formed by two tangents is measured by one *half the difference* between the intercepted arcs.

8. Find the number of degrees in the arc of the segment in which a $40°$ angle is inscribed.

9. Circumscribe a circle about a triangle whose sides are $12'$, $13'$, $5'$. Find the radius of the circle.

10. Circumscribe a circle about a triangle whose sides are 12, 9, 15. Find the radius.

11. A side of a triangle is $14''$. The angles adjacent to this side are $30°$ and $60°$, respectively. Circumscribe a circle about the triangle and find its radius.

12. The angle formed by two tangents is $44°$. Find the arcs intercepted.

13. The smaller of two arcs intercepted by two tangents drawn from the same point is $120°$. The chord of contact is $29'$. Find the length of tangent and the radius of the circle.

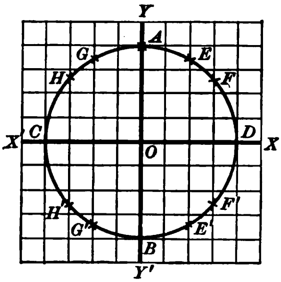
14. Two circles whose centers are O and O', respectively, intersect at A and B. OA is $40'$, $O'A$ is $9'$, and the distance between the centers is $41'$. Find length of the common chord AB (§ 192).

15. In problem 14, a tangent to each circle is drawn through point A. Find the angle between the tangents.

266. Every curve studied under § 235 had the same general form. Each had one bend or turning point, and two branches extending to an infinite distance from the turning point. Every quadratic in one unknown or variable produces a curve of this kind. It is a parabola. It is the path a stone takes when thrown into the air. It is also the path along which some comets move.

It is possible to produce closed curves by means of quadratics, but they must be quadratics in *two unknowns*. We will try some of this type.

Ex. 1. Find the graph of $x^2+y^2=16$.



x	y
0	± 4
± 2	$\pm 2\sqrt{3} = \pm 3.464$
± 3	$\pm \sqrt{7}$
± 4	0

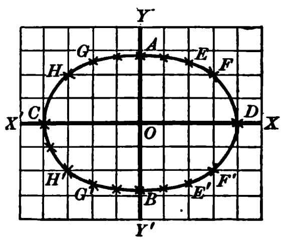
Since x and y both appear in the equation in the second degree only, the substitution of the same positive or negative number for either variable will produce the same absolute value for the other variable. For example, when $x=+2$, $y=+3.464$. That is, when $x=+2$ or -2, we get points E, E', G, G'.

Note that the points of this curve are symmetrically situated with respect to the origin O. Also the curve does not extend farther from O than ± 4. The curve is a circle, as is every curve of the form $x^2+y^2=a^2$, where a is the radius and the coefficients of x^2 and y^2 are each 1 and positive.

If the coefficients of x^2 and y^2 are positive and not equal, $AB \neq CD$, and the curve becomes an ellipse.

Ex. 2. Find graph of $x^2+2y^2=16$.

x	y
0	$\pm 2\sqrt{2} = \pm 2.828$
± 1	$\pm \sqrt{7.5} = \pm 2.738$
± 2	$\pm \sqrt{5} = \pm 2.449$
± 3	$\pm \sqrt{3.5} = \pm 1.87$
± 4	± 0
± 5	$\pm\sqrt{\frac{-9}{2}}$ (imaginary)



267. It is evident that a straight line can cut the curve in § 266 in but two points. Suppose we intersect a curve such as $x^2+y^2=16$ (Ex. 1, § 266), by a line $x+y=3$. In §§ 46–48,

we found that in simultaneous equations it was necessary to eliminate one of the variables, or unknowns. In quadratic equations some special method of elimination is often necessary.

First Method (Substitution)

Ex. 1. Solve $x^2 + y^2 = 16,$ (1)

$$x + y = 3. \quad (2)$$

From (2), $y = 3 - x.$ (3)

Substitute (3) in (1),

$$x^2 + (3 - x)^2 = 16. \quad (4)$$

$$2x^2 - 6x + 9 = 16.$$

$$2x^2 - 6x = 7$$

$$\underline{2 \cdot 4}$$

$$(4x)^2 - 2(4x)6 = 56.$$

$$(4x)^2 - 2(4x)6 + 36 = 92.$$

$$4x - 6 = \pm\sqrt{92}.$$

$$4x = \pm\sqrt{92} + 6$$

$$= \pm 9.59^+ + 6$$

$$= -3.59 \text{ or } +15.59.$$

$$x = -\frac{3.59}{4} \text{ or } +\frac{15.59}{4}$$

$$= -.897 \text{ or } 3.897.$$

$$y = 3 - x = 3 + .897 \text{ or } 3 - (+3.897)$$

$$= 3.897 \text{ or } -.897.$$

Second Method

$$x^2 + y^2 = 16, \quad (1)$$

$$x + y = 3. \quad (2)$$

Square (2), $x^2 + 2xy + y^2 = 9$

Subtract (1), $\underline{x^2 \qquad + y^2 = 16}$

$$2xy \qquad = -7 \quad (3)$$

Subtract (3) from (1), this gives the square of $x - y$.

$$x^2 - 2xy + y^2 = 23.$$

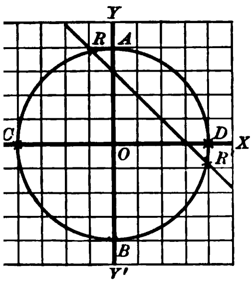
$$x - y = \pm\sqrt{23}. \quad (4)$$

Eliminate x and y between (2) and (4).

$$x + y = 3 \qquad (2)$$
$$x - y = \pm\sqrt{23} \qquad (4)$$
$$2x \quad = 3 + \sqrt{23} \text{ or } 3 - \sqrt{23}.$$
$$2x = 3 + 4.795 \text{ or } 3 - 4.795.$$
$$x = 3.897 \text{ or } -.897,$$
$$y = -.897 \text{ or } 3.897.$$

That is, these two equations have the solution,

$$x = 3.897,\ y = -.897,$$
1 $$x = -897,\ y = 3.897.$$



Note that the y-value obtained from using $x = 3.897$ goes with that value of x and the y-value obtained from $x = -.897$ goes with $x = -.897$.

Plotting equations (1) and (2) on the same axes, we have the following graphical representation of their solution:

Here R and R', the points of intersection of the curve and the line correspond to the algebraic solutions obtained by both the first and second methods.

EXERCISE 97

Plot the following equations, comparing in each instance the ;ebraic solution and the coördinates of the points of intersec- n in the graphs:

1. $\begin{cases} x^2 + y^2 = 4, \\ x + y = 2. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 25, \\ x - y = 5. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 25, \\ 3x + 4y = 25. \end{cases}$

In example 3 is the line a secant to the curve? If radii e drawn from the center of the circle to the points of interction of the curve with the line, how large is the angle *tween the radii?*

4. $\begin{cases} x^2+y^2=25, \\ 4x+3y=25. \end{cases}$

5. $\begin{cases} x^2+y^2=36, \\ x-2y=4. \end{cases}$

6. $\begin{cases} x^2+y^2=169, \\ 5x+12y=169. \end{cases}$

7. $\begin{cases} y^2=4x-8, \\ x-y=2. \end{cases}$

8. $\begin{cases} x^2=4y-8, \\ x-y=-2. \end{cases}$

9. $\begin{cases} y^2=4x, \\ x=4. \end{cases}$

10. An isosceles triangle is inscribed in the circle $x^2+y^2=25$. The base of the triangle is formed by the line $x=3$. Find the area and the angles of the triangle.

11. A triangle is inscribed in the circle $x^2+y^2=16$. One side is $x+y=4$, another side is $x-y=4$. Find the base, altitude, area, angles.

12. A triangle is inscribed in the circle $x^2+y^2=25$. One side of the triangle is $4x-5y=-20$, another side is $4x+5y=-20$. Find the third side, the area, the angles.

13. A square is inscribed in $x^2+y^2=64$. Find its area and its diagonal.

14. Find the points where the lines $x-y=-4$ and $x+y=-4$ intersect the curve $y^2=8x+16$. Find the area of the triangle formed by these lines and the Y-axis.

15. a and b are adjacent sides of a parallelogram circumscribed about a circle. Find the ratio of a to b.

16. A quadrilateral $ABCD$ is inscribed in a circle. Show that $\angle A$ is the supplement of $\angle C$.

17. Three angles are inscribed in a segment. Show that the bisectors of these angles meet in a point.

18. $ABCD$ is an inscribed quadrilateral. $\angle A=49°$, $\angle B=64°$. Find the other angles.

19. $ABCD$ is an inscribed quadrilateral. $\angle D=120°$, arc $BC=151°$, arc $AD=68°$. Find angles A, B, C.

20. Construct a tangent to a circle at a given point on the circumference.

21. If a quadrilateral is circumscribed about a circle, the sum of two opposite sides is equal to the sum of the other two sides.

22. An equilateral triangle is inscribed in a circle. Show that a tangent drawn through the vertex is parallel to the base of the triangle.

23. Construct a circle tangent to three lines which do not pass through a point. How many circles satisfy this condition?

24. The radius of a circle is 12″. Draw a chord in this circle 7″ long and parallel to a given line.

25. In example 24 draw the chord perpendicular to the line.

26. Inscribe a triangle in a circle whose radius is 8″ such that the angles of the triangle are in the ratio 1, 2, 3.

27. Inscribe a triangle in a circle of radius 20″ such that the sides shall be proportional to 3, 4, 5.

28. Through a point $x = 6''$, $y = 8''$ in the circle $x^2 + y^2 = 100$, draw a chord 16″ long. How far is the point from the center? How far is the chord from the center? Could this chord be drawn through $x = 3''$, $y = 4''$?

29. An isosceles triangle is inscribed in $x^2 + y^2 = 169$. One leg of the triangle is formed by the line $5x + y = 13$. Find the area of the triangle.

30. A triangle whose sides are in the ratio 3, 4, 5 is inscribed in a circle. The perpendicular drawn from a vertex to the longest side of the triangle is 12″. Find the sides of the triangle and the radius of the circle.

31. Two unequal circles are tangent externally at T. Through T lines AA' and CC' are drawn intersecting the first circumference at A and C and the other circumference at C' and A'. Prove arcs AC and $C'A'$ have the same number of degrees and that the triangles formed are similar.

32. Two unequal circles intersect at M and K. Through these points M and K lines AB and DC are drawn intersecting the circumferences at A, B, C, and D. Draw AD and BC and prove that $ABCD$ is a trapezoid.

33. A number is composed of two digits which differ by 3. If the square of the number is added to the square of the number formed by reversing the digits, the result is 3329. Find the number.

34. The difference between the numerator and denominator of a fraction is 2, and the sum of the fraction and its reciprocal is $\frac{290}{143}$. Find the fraction.

35. The difference between the numerator and denominator of a certain fraction is 7. The difference between the fraction and its reciprocal is $1\frac{41}{120}$. Find the fraction.

36. The sum of a number and its reciprocal is $\frac{169}{60}$. What is the number?

37. The sum of a number and its reciprocal is $\frac{5}{11}$. What is the number?

38. The difference between a number and its reciprocal is $\frac{5}{11}$. What is the number?

39. Find three consecutive numbers such that their product divided by their sum is 56.

40. The legs of a right triangle are in the ratio $\frac{5}{12}$. The square on the hypotenuse is 1521. Construct the triangle.

41. Two similar rectangles have their homologous sides in the ratio 2 to 3. The length and breadth of the smaller rectangle are in the ratio $\frac{7}{2}$. The area of the larger rectangle is 504. Find the dimensions of the smaller rectangle.

42. The area of a trapezoid $ABCD$ is 108. AB is 8 more than the altitude; DC is 2 less than the altitude. Find the dimensions. Explain your negative results.

43. In a trapezoid $ABCD$, $\angle A$ is $45°$; $\angle B$ is $60°$; AD is $9\sqrt{2}$; the area is $\dfrac{297+27\sqrt{3}}{2}$. Find the dimensions.

44. A right triangle ABC, right-angled at B, has one leg $7'$ longer than the other, and the hypotenuse $8'$ longer than the shorter leg. Find the dimensions. Explain both solutions.

45. A right triangle ABC, right-angled at B, has one leg $7'$ longer than the other. The longer leg is two thirds the sum of the hypotenuse and the shorter leg. Find the lines and angles of the triangle.

46. The area of a triangle ABC is 240. BC is $26'$, and the altitude upon AB bisects the base. Solve the triangle.

47. A rectangle is $18'' \times 24''$. How much longer must the length be to increase the diagonal $4''$? Explain negative results.

48. A rectangle is $18'' \times 24''$. This is to be increased in length and breadth until the diagonal produced is increased $4''$. What must the new length be? Explain negative results.

49. A city playground is rectangular, a walk $1420'$ long runs diagonally across the grounds, and the distance around the grounds is $3976'$. Find the dimensions of the grounds.

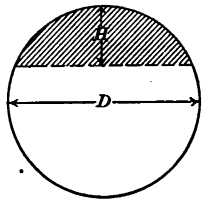
50. A rectangular piece of sheet metal is $6''$ longer than it is wide. A square $4''$ on a side is cut from each corner of the metal, and the balance folded up to form a box which contains 1408 cubic inches. Find the dimensions of the sheet of metal.

51. A rectangle is inscribed in $x^2 + y^2 = 676$. The area of the rectangle is $a^2 + 12a - 160$. The longer side is 10 from the center. Find dimensions.

52. A rectangle is inscribed in $x^2 + y^2 = 289$. Call a the altitude and b the base. The sum of the base and altitude is 23, and the area of the rectangle is 120. Find dimensions.

Supplemental Applied Mathematics *

The area of a circular segment is used in finding the steam space in boilers and also in finding the quantity of liquid contained in partially filled circular tanks.



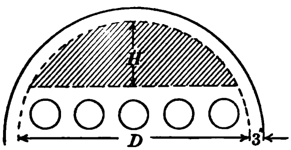
To find the area of a circular segment, two things must be known. These are often the height of the segment, H, and the diameter, D; or the degrees in the arc and the length of the chord; or the degrees of the arc and D; or the length of the chord and D.

The formula for the shaded area in the figure is

$$\frac{4}{3}H^2\sqrt{\frac{D}{H}-.608}.$$

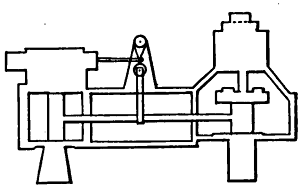
1. Find the area of a circular segment when $D = 12''$, $H = 3''$. Solve by the above formula, then by geometry and trigonometry, and compare results.

2. A horizontal oil tank of circular cross section is 10′–6″ long, $D = 2'$–$6''$. This tank is filled to within 6″ of its capacity. That is, $H = 6''$. How much oil is in the tank?



3. In a 60″ boiler H is 20″, $D = 54''$. Find area of shaded portion.

4. Water cylinders of duplex double-acting pump are 6″ diameter by 8″ stroke. Compute discharge in gallons per minute, running at 40 strokes per minute on each side.



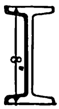
5. The volume of the unfilled portion of a horizontal tank of circular cross section is 36 cubic feet. The H of the unfilled portion is 8″. The length of the tank is 10′–0″. Find D.

* The Supplemental Applied Mathematics for girls is found on page 337.

6. The cylinder of a steam engine is 8″ in diameter. The connecting rod of the piston is attached to the drive wheel 10″ from the center of the wheel. What is the rubbing surface of the cylinder?

7. A pipe line terminates in a vertical cylindrical tank 30′ in diameter. The pipe is 6″ in diameter and the pump forces the oil through the pipe line at the rate of two barrels a minute. How many inches will the surface of the oil in the tank rise in 3 hours?

8. A coil of $\frac{1}{8}$″ steel wire weighs 110 pounds. How many feet of wire in the coil?



9. A cross section of a steel I beam is 8″ entire height, 6″ between the flanges, $\frac{3}{4}$″ thickness in its narrowest part, and the flange width is 3″. Find its weight per lineal foot. Consider the forms as trapezoids and rectangles.

10. Find the per cent of error in using $3\frac{1}{7}$ for π in place of 3.1416.

Weight of Metals

	Cu. In. Lb.	Cu. Ft. Lb.
Wrought iron	.28	480
Cast iron	.26	450
Steel (soft)	.28	480
Cast steel	.29	496
Brass	.30	520
Copper	.32	550
Lead	.41	711

11. Find the weight of a piece of shafting 20 feet long and 2″ diameter. What will be the weight of a piece of the same length and 1″ diameter?

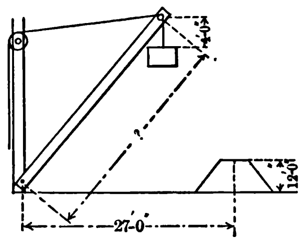
12. A safety valve is loaded with 7 cast iron weights 10″ *diameter × $1\frac{1}{2}$″ thick.* Find the total weight on the valve.

13. Show that the distance across the corners of a square nut = 1.414 × distance across the flats.

14. If the safety valve in example 12 is $2\frac{1}{2}''$ in diameter, find the steam pressure per square inch necessary to just balance the weights.

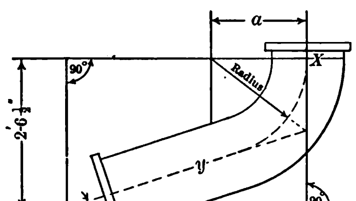
15. A cast steel cylinder is $42''$ inside diameter, $4'$–$6''$ long, and $1\frac{1}{4}''$ thick. Find weight of it.

16. 50 studs $1'' \times 2\frac{1}{4}''$ are to be cut from cold rolled steel. Find length and weight of bar necessary, allowing $\frac{1}{8}''$ for cutting off.



17. How long must the boom be to land load on the 12-foot pedestal, allowing $4'$ clearance at the end for ropes and pulleys?

18. A pipe enters a machine at an angle of 22° 30′. y is the center line of the pipe. With dimensions as per figure, turn the pipe as indicated and find the center of curvature and dimension a. That is, find the center and radius of an arc passing through point X and tangent to line y.



19. The strength of a wooden beam supported at the ends is inversely proportional to the length in feet and directly proportional to the product of the width in inches by the square of the depth in inches. How much stronger is a beam $14' \times 8'' \times 2''$ than a beam of the same material $12' \times 6'' \times 2''$?

20. How much stronger is an oak beam $12' \times 8'' \times 2''$ than an oak beam $18' \times 8'' \times 2''$?

The *strength* of a beam varies with the kind of wood used. The difference in strength is determined by a *constant factor* for each kind of wood. The constant is determined by testing a piece of wood $1' \times 1'' \times 1''$.

The *breaking strength* is $\frac{k \cdot b \cdot d^2}{L}$.

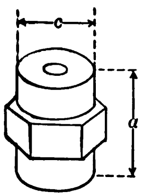
The constant, k, for white pine is about 300 pounds.

The constant, k, for oak is about 1850 pounds.

In estimating strength of beams a *safety factor* 10 is added for live loads and 5 for dead loads.

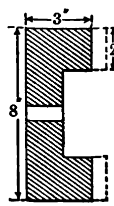
21. A hall floor $20' \times 100'$ is supported by $3'' \times 10''$ pine joists placed on edge. Find distance of the joists apart (O. C.), allowing for a live load. (People in a closely packed house average about 140 pounds per square foot of floor space.)

22. Find distance O. C. in the above problem if the joists are oak and are $2'' \times 8''$.



23. Given a sleeve of length a. The hexagonal part is one third the total length. The diameter c of the cylindrical part is one half the total length. The diameter of the hole is $\frac{1}{2}c$. The distance across the flats is $3\frac{3}{8}''$ and is $2\frac{1}{4}$ times the diameter of the hole. Find all dimensions and the weight when made of brass.

24. A flywheel is $8''$ in diameter and has not sufficient weight. It is not possible to increase the diameter of the wheel, so 20 pounds of steel is added to the width of the rim, which is now $3''$ wide by $2''$ thick. Required the width and thickness of the part added.



25. If the diameter in problem 24 could be increased $\frac{1}{2}''$, how much would be added to the weight?

26. If the diameter were increased $\frac{3}{4}''$ and $\frac{1}{2}''$ were added to the width of the rim, by how much would the weight be increased?

27. The altitude of a right triangle is $\frac{1}{8}''$; the base is π''. Find the ratio of the altitude to the base.

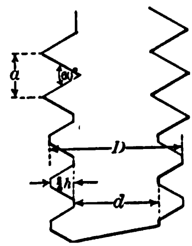
NOTE. A screw thread is the curve made by the hypotenuse of a right triangle wrapped about a cylinder. The base of the right triangle is $2\pi R$ or πD, where R is the radius and D the diameter of the cylinder. The pitch of the screw is the distance between the threads, measurements being taken parallel to the axis of the cylinder. The curve which the hypotenuse makes is called the *helix*.

28. The diameter of a screw is 3″, the number of threads to the inch is $3\frac{1}{2}$. Find the ratio of the pitch to the base of the triangle used in forming the helix.

29. The diameter of a screw is 6″ with $2\frac{1}{2}$ threads to the inch. Find the ratio of the pitch to the base of the triangle.

30. How far will the screw in example 29 advance in a quarter of a turn? In half a turn?

In a U.S. standard thread the included angle of the cut is 60°. A vertical section of the bolt would then show a series of equilateral triangles arranged along the sides of the bolt were it not that the top and lower part of each triangle is cut off so that only $\frac{3}{4}$ the altitude h remains. a, the distance between the threads, is the pitch, and is the base of the equilateral triangle. Hence, if there were 5 threads to the inch, a would be $\frac{1}{5}''$ and h would be $\frac{1}{2}\left(\frac{1}{5}\right)\sqrt{3}''$. D is the diameter of the bolt, and d, the part of the diameter left after the cut is made, is the root diameter of the screw.



31. The diameter of a bolt is 1″. A U.S. standard screw thread is cut on this bolt, 8 threads to the inch. Find the root diameter of the screw.

32. In a U. S. standard screw there are $2\frac{1}{2}$ threads to the inch. The diameter of the bolt is 6″. Find the slope of the hypotenuse of the right triangle which develops the thread.

33. The pitch of a U. S. standard screw is $\frac{1}{4}''$. How far will the screw advance in a 60° turn? In 120°? In 180°? In 360°? In 270°?

34. A 3-inch bolt has $3\frac{1}{2}$ threads to the inch. Find the root diameter of the screw.

35. To what diameter must you upset a 1″ rod so that the root diameter of the screw will be 1″? This U. S. screw has 6 threads to the inch.

36. One acute angle of a right triangle is $2° 16.74'$. The base equals the perimeter of a 1″ bolt. Will this triangle develop a U. S. standard screw thread? What is the pitch of the screw?

37. A small cylinder head $4\frac{1}{8}''$ diameter is held on by four $\frac{5}{8}''$ studs having 11 threads to the inch. Find the pull on the studs per square inch of area when the pressure in the cylinder is 125 lb. per square inch.

38. An I beam, each end of which rests on a stone pier, carries a load of 6 tons at the center of the beam. The safe load for the stone used is 250 pounds per square inch. How much bearing surface must each end of the beam have?

CHAPTER XVI

Applications of the Circle. Proportionals

THEOREM LXXIII

268. *If two chords intersect, the product of the segments of one is equal to the product of the segments of the other.*

Draw a circle; draw chord AB and chord CD intersecting at P. We have

Given chords AB and CD intersecting at P.

To prove $AP \cdot PB = CP \cdot PD$.

Proof. 1. Draw AC and BD.

2. Use § 182.

(The proof is left to the student.)

THEOREM LXXIV

269. *If through a fixed point without a circle a secant and a tangent are drawn, the tangent is a mean proportional between the whole secant and its external segment.*

Draw a circle. Through an external point A draw tangent AT, and secant AC cutting the circumference at B. We have

Given tangent AT and secant AC.

To prove $\dfrac{AB}{AT} = \dfrac{AT}{AC}$.

Draw chords TB and TC. Compare $\triangle$ ATB and ATC.

(The proof is left to the student.)

EXERCISE 98

1. If from any point in the circumference of a circle a perpendicular be drawn to a diameter of the circle, the perpendicular is a mean proportional between the segments of the diameter.

Let AB be the diameter and PD the perpendicular, P being in the circumference. Draw PA and PB.

2. If from any point in the circumference of a circle a perpendicular be drawn to the diameter, the chord joining the point to either extremity of the diameter is a mean proportional between the whole diameter and the adjacent segment.

3. Draw two circles tangent internally; draw a chord of the larger circle tangent to the smaller; through the points of tangency draw a chord of the larger circle. Connect the extremities of these chords to form a quadrilateral; show that there are two pairs of similar triangles in the figure.

4. Show that if the diagonals of any inscribed quadrilateral are drawn, the quadrilateral is decomposed into two pairs of similar triangles.

5. Given two concentric circles and a chord AB of the larger circle such that its intersection points, R and S, with the smaller circumference are trisection points of the chord AB. If O is the centre of the circle, $\angle ROS$ is 60°, and $AB = 60'$, find radius of larger circle and distance of AB from center.

6. In example 5 find the radius of each circle if $ROS = 40°$.

7. In a circle whose radius is $24'$ a chord is drawn such that the angle formed by radii drawn to the extremities of the chord is 75°. Find the length of the chord.

8. A chord $16'$ long subtends an arc of 82°. Find the distance of the chord from the center of the circle.

9. Two parallel chords are $8'$ and $16'$, respectively, in length and are $4'$ apart. Find the radius of the circle.

10. Two chords AB and CD intersect at S in such a manner that chord CD is divided at S in the ratio $\frac{1}{2}$. AB is 16 and CD is 12. Find the product of the segments of AB.

11. In a circle whose radius is 28″ a chord 24″ long intersects a second chord. One segment of the first chord is 4″ and one segment of the second chord is 5″. Find the length of the second chord.

12. In a circle $x^2+y^2=144$ a chord is drawn making an angle of 45° with the diameter of the circle. If the diameter is AB, the chord CD, their intersection P, and PD is 8, find CD and the segments of AB formed by a perpendicular drawn from D to AB. Is there more than one solution? Illustrate.

13. In the circle $x^2+y^2=100$ a chord CD intersects a diameter AB at an angle of 30°. If P is the point of intersection and PD is 4, find CD and the segments of AB made by a perpendicular drawn from D to AB.

14. Through P a tangent PT and a secant PD are drawn. The secant cuts the circle at C. PC is 4 and CD is 12. Find PT.

15. Given a circle $x^2+y^2=64$ and a tangent drawn from a point $x=17$, $y=0$. Find the length of the tangent. Find the angle between the tangent and the X-axis.

16. A tangent is 12″ long and through the extremity of the tangent a secant is drawn whose internal segment is 10″. Find the length of the secant.

17. Given a circle whose diameter is TM. At T a tangent is drawn, and from M a secant is drawn meeting the tangent at P and forming with the tangent an angle of 30°. PM is 48″. Let K be the point where PM cuts the circumference. Find PT, PK, and the diameter of the circle.

18. Solve the above problem when $\angle P=35°$.

19. Two unequal circles are tangent externally. From any point in their common tangent drawn through their point of contact a secant is drawn to each circle. Show that the product of one secant by its external segment is equal to the product of the other secant by its external segment.

20. A parallelogram is circumscribed about a circle whose diameter is 18″. One angle of the parallelogram is 135°. Find its area.

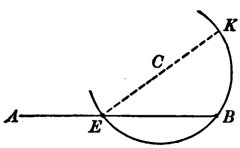
21. Two chords intersect, the segments of one are 7 and 14, one segment of the second is 5. Find the other segment.

22. The segments AP and PB of a chord AB are 5 and 12, respectively. Any chord whose segments are x and y is drawn through P. Find the area of a rectangle whose base and altitude are this x and y.

23. Through a point P in a circle whose radius is 12 a chord AB is drawn. The segment $AP = 6$, and $PB = 12$. Other chords whose segments are x, y; x_1, y_1; x_2, y_2; etc., are drawn through P. Show that the product xy, x_1y_1, x_2y_2, etc., is constant.

24. At a point T on the circumference of a circle whose radius is 28 a tangent PT is drawn. PT is 12. Show that if any secant is drawn through P, the whole secant and its external segment are a pair of factors of 144.

25. To construct a perpendicular at the end of a line without producing the line.



Let AB be the given line.

With any point C, between A and B, but without the line, as a center, and radius CB, describe an arc intersecting AB at E.

Draw diameter EK.

KB is the required perpendicular.

(The proof is left to the pupil.)

PROBLEM

270. *To construct a fourth proportional to three given lines.*

Draw three lines a, b, and c. We have

Given lines a, b, and c.

To construct a fourth proportional to a, b, and c.

Construction. 1. Construct an angle KLM.

2. On LK take $LC = a$, and $CD = b$.

3. On LM take $LR = c$, and draw CR.

4. Draw $DS \parallel CR$ meeting LM at S.

5. RS is the fourth proportional,

$$\text{for } \frac{a}{b} = \frac{c}{RS}. \quad (\S\S\ 179,\ 175.)$$

If b and c are equal, RS is the third proportional to a and b.

PROBLEM

271. *To construct a mean proportional between two lines.*

Given lines a and b.

To construct a mean proportional between them.

Construction. 1. Describe a circumference on $a + b$ as diameter.

2. On this diameter take $AB = a$ and $BC = b$.

3. At B erect a perpendicular to AC meeting the circumference at K.

4. KB is the mean proportional.

(The proof is left to the pupil.)

The construction of third, fourth, and mean proportionals is useful in many kinds of geometric drawing, especially in constructing equivalent and similar figures.

PROBLEM

272. *To divide a line in extreme and mean ratio.**

Given line AB.

To divide AB into extreme and mean ratio.

1. With a radius $BC = \frac{1}{2} AB$ describe a circle tangent to AB at B, C being the center of the circle.

2. Draw a secant passing through A and C cutting the circumference at E and G.

3. With A as center and AE and AG as radii describe arcs cutting AB and AB produced at E' and G', respectively.

4. E' and G' are the required points of division.

Proof. 1. $\frac{AG}{AB} = \frac{AB}{AE}$ or $\frac{AG'}{AB} = \frac{AB}{AE'}$. (§ 269.)

2. Using division, $\frac{AG' - AB}{AB} = \frac{AB - AE'}{AE'}$.

3. But $AB = 2\,BC = EG$.

4. Then $\frac{AG' - AB}{AB} = \frac{AB - AE'}{AE'}$ becomes $\frac{AE'}{AB} = \frac{E'B}{AE'}$.

5. Inverting 1 and using composition, we have

$$\frac{AG' + AB}{AG'} = \frac{AE' + AB}{AB}.$$

Or $\frac{G'B}{AG'} = \frac{AG'}{AB}$. $(AE' + AB = AE + EG = AG.)$

This construction is used in inscribing a decagon in a circle.

EXERCISE 99

1. Construct a mean proportional between 9 and 16.

2. Construct a mean proportional between 1 and 3. Is this equivalent to drawing a line $\sqrt{3}$ long?

* A line is divided in extreme and mean ratio when one segment is a mean proportional between the whole line and the other segment. The line may be divided either externally or internally.

3. Find a line equal to $\sqrt{2}$; $\sqrt{5}$; $\sqrt{6}$.

4. Find a mean proportional between $\sqrt{2}$ and $\sqrt{3}$.

5. Construct a line which is a mean proportional between $\sqrt{2}$ and 3.

6. Divide a 12″ line in extreme and mean ratio.

7. If two secants are drawn through the same point without a circle, the product of one and its external segment is equal to the product of the other and its external segment.

8. A tangent has the same length as the diameter of a circle. How far is the extremity of the tangent from the center of the circle?

9. A tangent and secant drawn from the same point are respectively 18″ and 36″. The secant passes through the center of the circle. Find the diameter.

10. AB is the common chord of two equal circles. Through A a tangent to each circle is drawn meeting the circumferences in K and L. Show that $KB = BL$. Is KBL ever a straight line? If so, when?

11. Let O be the center of a circle and AB a chord perpendicular to a diameter meeting the diameter at K. Draw a diameter from A. Draw BD perpendicular to this diameter. Show that △ AOK and ADB are similar.

12. Two circles are tangent internally, the diameter of one circle being the radius of the other. Show that any chord of the larger circle drawn through the point of contact is bisected by the circumference of the smaller circle.

13. Construct a fourth proportional to lines whose lengths are 8″, 5″, 11″, respectively.

14. Construct a third proportional to lines whose lengths are 8, 5, respectively.

15. Construct a fourth proportional to lines whose lengths are $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, respectively.

16. Construct a third proportional to $\sqrt{3}$, $\sqrt{5}$.

Problem

273. *To construct a square equivalent to the sum of two given squares.*

Given two squares A and B.

To construct a square $\approx A + B$.

Construction. 1. Construct a right triangle whose base and altitude are the sides of A and B, respectively.

2. The hypotenuse of this triangle will be the side of the required square. (§ 193.)

Problem

274. *To construct a square equivalent to a given parallelogram.*

(The mean proportional between the base and altitude of the parallelogram is the side of the required square. The construction and proof are left to the student.)

Problem

275. *To construct a square equivalent to a given triangle.*

(The side of the required square is the mean proportional between the base and one half the altitude of the given triangle.)

EXERCISE 100

1. Construct a square equivalent to the sum of three squares whose sides are respectively a, b, c.

2. The bases of a trapezoid are 14″ and 18″; the altitude is 4″. Construct an equivalent square.

3. The bases of a trapezoid are 14″ and 18″; the altitude is 4″. Construct an equivalent isosceles triangle.

4. The sides of a parallelogram are 14″ and 18″; one angle is 64°. Construct an equivalent square.

5. Two sides of a triangle are 16″ and 12″, respectively; the *included angle is 37°*. Construct an equivalent square.

6. The sides of a triangle are respectively 6″, 8″, 11″. Construct a triangle which has the same angles as the given triangle and whose area is 4 times as great.

7. Construct a square equivalent to the difference between two given squares.

8. Construct a triangle similar to that given in example 6, and whose area is three times as great.

CHAPTER XVII

Regular Polygons, Areas, Loci

THEOREM LXXV

276. *A circle can be circumscribed about any regular polygon.* (§ 147.)

Draw a regular hexagon whose vertices are A, B, C, D, E, F. We have

Given a regular polygon $ABCDEF$.

To prove that a circle may be circumscribed about it.

Proof. 1. Pass a circumference through A, B, C (§ 257), having center O. Draw radii OA, OB, OC. Draw OD.

2. In $\triangle$s OAB, OCD, $OB = OC$; $AB = CD$. (?)

3. Prove the angles included by these sides are respectively equal. (Remember $\angle ABC = \angle BCD$.)

4. If $\triangle OAB = \triangle OCD$, OD is a radius.

5. It may then be shown that OE and OF are radii.

277. AB, BC, etc., are equal chords and are therefore equally distant from the center O. Then if O is taken as center, and a perpendicular from O to AB as radius, a circle may be inscribed in $ABCDEF$. Hence, *a circle may be inscribed in a regular polygon.*

278. The center of the circumscribed and inscribed circles is the **center of the polygon.**

The **apothem** is the radius of the inscribed circle.

The **radius** is the radius of the circumscribed circle.

The **angle at the center** of a regular polygon is the angle *formed by two radii* drawn to the extremities of any side.

279. The sum of the angles about O is four right angles; hence *any angle at the center of a regular polygon is equal to four right angles divided by the number of sides.*

Theorem LXXVI

280. *If a circumference is divided into equal arcs, the chords of these arcs form a regular polygon.*

Draw a circumference and divide it into six equal arcs at A, B, C, D, E, F. Draw chords AB, BC, CD, etc. We have

Given circumference ACE divided into six equal arcs and AB, BC, CD, etc., drawn.

To prove $ABCDEF$ a regular polygon.

Proof. If the polygon is regular, it must be equilateral and equiangular. (The proof is left to the student.)

281. In the figure of § 280, draw tangents to the circle at A, B, C, D, E, F. Let these tangents intersect, forming the polygon $HIJKLM$. Show that this polygon is regular; that is, *if a circumference is divided into any number of equal parts, tangents drawn at the points of division form a regular polygon.*

EXERCISE 101

1. The polygon in § 280 was a regular hexagon. Inscribe a regular triangle.

2. Given a regular inscribed hexagon, to inscribe a regular dodecagon.

3. Given a circle, to circumscribe a regular dodecagon.

4. Find the angle at the center of a regular dodecagon.

5. Inscribe a square in a circle.

6. Inscribe a regular octagon in a circle.

7. Show that an equilateral polygon inscribed in a circle is regular.

8. Find the angle at the center of a regular octagon.

9. Two regular polygons of the same number of sides are similar.

THEOREM LXXVII

282. *The perimeters of two similar polygons are in the same ratio as their radii, or as their apothems.*

Draw two similar polygons $ABCD$ etc., and $A'B'C'D'$ etc., having centers O and O', respectively. We have

Given two similar polygons ABC etc., and $A'B'C'$ etc.

To prove $$\frac{ABCD \text{ etc.}}{A'B'C'D' \text{ etc.}} = \frac{OB}{O'B'} = \frac{r}{r'},$$

where r and r' are apothems.

(Let $OB = R$ and $O'B' = R'$.)

Proof. 1. The triangle formed by R, r, and BC is similar to the triangle formed by R', r', and $B'C'$.

2. Then, $$\frac{R}{R'} = \frac{r}{r'} = \frac{BC}{B'C'}.$$

3. (Then use § 191.)

4. Hence,

283. Let K be the area of the first polygon in § 282 and K' the area of the second.

From § 211, $$\frac{K}{K'} = \frac{\overline{BC}^2}{\overline{B'C'}^2}.$$

From step 2, § 282, $$\frac{BC}{B'C'} = \frac{R}{R'} = \frac{r}{r'}.$$

Then, $$\frac{K}{K'} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}.$$

Or, *the areas of two similar polygons are in the same ratio as the squares of their radii, or as the squares of their apothems.*

THEOREM LXXVIII

284. *The area of a regular polygon is equal to one half the product of its perimeter and apothem.*

From O, the center of a regular polygon of n sides, draw the radii and apothem; n equal triangles are formed whose common altitude is r. The proof is left to the student.

EXERCISE 102

1. In a regular hexagon find the apothem in terms of R.

2. Find the apothem of an equilateral triangle. What relation is there between the altitude and the apothem?

3. Draw an isosceles right triangle whose hypotenuse is 2. Find the cosine of the acute angle. What is the sine?

4. Inscribe the triangle in example 3 in a circle and find the side of the square inscribed in the circle.

5. In a circle whose radius is R, find the side of the inscribed square.

6. A triangle whose area is 96 square inches is inscribed in a semicircle whose radius is 16″. Construct the triangle.

7. A triangle whose area is 96 square inches is inscribed in a semicircle whose radius is 8″. Draw to a scale. Could a triangle so inscribed be equiangular?

8. One side of a regular hexagon is 5. Find the radius of the inscribed circle.

9. One side of a regular hexagon is $5\sqrt{3}$. Find the radius of the inscribed circle.

10. The diameter of a circle is 28″. Find the area of the inscribed regular hexagon.

11. In the circle in example 10, find the area of the inscribed equilateral triangle.

12. In a circle whose radius is R, find the ratio of the areas of the inscribed equilateral triangle and the inscribed regular hexagon.

13. In a semicircle whose radius is R, find the altitude of the inscribed right triangle of the greatest area.

14. In a circle whose radius is R, find the ratio of the area of the circumscribed regular hexagon to that of the inscribed regular hexagon.

15. Find a side of an equilateral triangle inscribed in a circle of radius R.

Problem

285. *To inscribe a regular decagon in a circle.*

Let CB be the radius of the circle and C the center. Describe the circle. We have

Given a circle with center C and radius CB.

To inscribe a regular decagon.

Construction. 1. Divide CB internally in extreme and mean ratio at A.

2. Then,
$$\frac{CB}{CA}=\frac{CA}{AB}.$$

3. CA is the side of the required decagon.

Proof. 4. Draw chord $BD=CA$. Draw AD and CD.

5. Since $CA=BD$, 2 becomes
$$\frac{CB}{BD}=\frac{BD}{AB}.$$

6. In $\triangle$ BDC and BDA, $\angle B=\angle B$ and by 5 the including sides are proportional.

7. Hence, $\triangle$ BDC and BDA are similar, and BDA is isosceles. (§ 186.)

8. Since $CA=BD=DA$, $\angle C=\angle ADC$.

9. $\angle B=\angle BAD=\angle C+\angle ADC=2\angle C$.

10. In $\triangle CBD$, $\angle C + \angle B + \angle BDC = 180°$.

Or, $$\angle C + 2\angle C + 2\angle C = 180°.$$

Or, $$\angle C = \frac{2 \text{ rt. } \angle s}{5}, \text{ or } \frac{4 \text{ rt. } \angle s}{10}. \quad (\S\ 279.)$$

286. If alternate vertices of a regular decagon are joined, a regular pentagon is formed.

An inscribed regular pentagon may also be constructed as follows:

1. Let O be the center of a circle. Diameter $AB \perp$ diameter CD; E the middle point of OB.

2. With E as center and ED as radius, describe an arc cutting AO at M.

3. DM is the side of the required pentagon.

287. In the proportion $\frac{CB}{BD} = \frac{BD}{AB}$, (§ 285, 5.)

$$CB = R;\ AB = R - CA = R - BD.$$

Substituting these values of CB and AB, we have

$$\frac{R}{BD} = \frac{BD}{R - BD}.$$

$$R^2 - R(BD) = (BD)^2.$$

Or, $$(BD)^2 + (BD)R = R^2.$$

Completing square,

$$4(BD)^2 + 4(BD)R + R^2 = 4R^2 + R^2,$$

$$2BD + R = R\sqrt{5},$$

$$BD = \frac{R\sqrt{5} - R}{2} = \frac{R(\sqrt{5} - 1)}{2},$$

the side of a regular decagon inscribed in a circle whose radius is R.

EXERCISE 103

1. The area of a regular hexagon inscribed in a circle is $24\sqrt{3}$. Find a side of a regular decagon inscribed in the same circle.

2. Find the ratio of the areas of a circumscribed square and an inscribed square.

3. Find the ratio of the areas of a circumscribed equilateral triangle and an inscribed equilateral triangle.

4. The radius of a circle is 16″. Find the area of the inscribed regular decagon.

5. Find the area of a regular decagon inscribed in a circle whose radius is 12.

6. The center of a regular hexagon bisects every line drawn through it, and terminating in the sides.

7. Find the area of a regular octagon inscribed in a circle whose radius is 18.

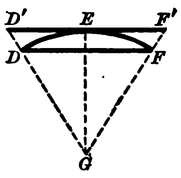
8. Find the radius of a circle inscribed in an equilateral triangle whose side is 8.

Theorem LXXIX

288. *If a regular polygon be inscribed in or circumscribed about a circle and the number of its sides be indefinitely increased,*

I. *Its perimeter approaches the circumference as a limit.*

II. *Its area approaches the area of the circle as a limit.*



Given K, k the areas, and P, p the perimeters of the circumscribed and inscribed regular polygons, respectively. Let G be the center and C the circumference of the circle; A the area of the circle.

To prove that if the number of sides of the polygons is indefinitely increased, P and p approach C as *a limit, and K* and k approach A as a limit.

Part I

Proof. 1. $D'F'$ is a side of the circumscribed polygon. GD' and GF' are drawn cutting the circumference at D and F, respectively. $D'F'$ is tangent at E. Draw GE. Draw DF.

2. DF is a side of the inscribed polygon.

3. $$\frac{P}{p}=\frac{GD'}{GD}=\frac{GD'}{GE}. \quad (\S\ 282.)$$

4. $$\frac{P-p}{p}=\frac{GD'-GE}{GE}. \quad (\S\ 174,\ 4.)$$

Or, $$P-p=\frac{p}{GE}(GD'-GE).$$

5. p is less than C, GE is constant

and $$GD'-GE<D'E. \quad (?)$$

6. Substituting these values in 4,

$$P-p<\frac{C}{GE}\cdot D'E.$$

7. GE is constant. If the number of sides of each polygon is indefinitely increased, the two polygons always having the same number of sides, $D'E$ grows smaller and approaches zero as a limit.

8. 6 then approaches

$$P-p<\frac{C}{GE}\cdot 0.$$

Or, $$P-p<0. \quad (?)$$

That is, the difference between P and p approaches 0.

9. Now, $P>C>p$.

10. Then $P-C$ approaches 0, and $C-p$ approaches 0.

11. Hence, the perimeter P and p approach C as their limit.

Part II

1. Similarly, $$\frac{K}{K}=\frac{(GD')^2}{(GE)^2}. \quad (\S\ 283.)$$

(Part II of the proof is left to the student.)

289. *The results of § 288 show that the apothem of the inscribed polygon approaches the radius of the circle as the number of sides is indefinitely increased.*

THEOREM LXXX

290. *The circumferences of two circles are in the same ratio as their radii.*

Given two circles whose circumferences are C_1 and C_2 and whose radii are R_1 and R_2, respectively.

To prove $$\frac{C_1}{C_2}=\frac{R_1}{R_2}.$$

Proof. 1. If regular polygons of the same number of sides are inscribed in these circles, we have

$$\frac{P_1}{P_2}=\frac{R_1}{R_2}.$$

2. If the number of sides of the polygons is indefinitely increased, we have $$\frac{P_1}{P_2}=\frac{C_1}{C_2}. \quad (\S\ 288.)$$

3. Hence, $$\frac{C_1}{C_2}=\frac{R_1}{R_2}.$$

In this theorem we are really regarding a circle as a regular polygon of an indefinite number of sides.

291. If both terms of $\frac{R_1}{R_2}$ are multiplied by 2, the proportion in § 290 becomes

$$\frac{C_1}{C_2}=\frac{D_1}{D_2}.$$

Or, *the circumferences of two circles are proportional to their diameters.*

This proportion may be written

$$\frac{C_1}{D_1}=\frac{C_2}{D_2}.$$

That is, the circumference is proportional to the diameter, or the ratio of the circumference to the diameter is a constant. By higher mathematics this constant has been found to be 3.14159+.

It is called π (the P of the Greek alphabet) and is usually considered 3.1416. For rough work $3\frac{1}{7}$ or $\frac{22}{7}$ is sometimes used.

Theorem LXXXI

292. *The area of a circle is equal to one half the product of the circumference and radius.*

Given area S, circumference C, radius R.

To prove $S = \frac{1}{2}CR$.

Proof. 1. Circumscribe a regular polygon about the circle.

2. $K = \frac{1}{2}PR$. (§ 284.)

3. Let the number of sides indefinitely increase.

4. (Use § 288.)

293. Since $\frac{C}{D} = \pi$, or $C = \pi D = 2\pi R$,

substituting this value of C in $S = \frac{1}{2}CR$, we have,

$$S = \tfrac{1}{2}\cdot 2\pi R\cdot R = \pi R^2.$$

In terms of D this becomes $S = \frac{1}{4}\pi D^2$.

Denoting the areas of two circles by S_1 and S_2,

$$\frac{S_1}{S_2} = \frac{\pi R_1^2}{\pi R_2^2} = \frac{R_1^2}{R_2^2}$$

$$= \frac{\frac{1}{4}\pi D_1^2}{\frac{1}{4}\pi D_2^2} = \frac{D_1^2}{D_2^2}.$$

That is, *the areas of circles are proportional to the squares of their radii, or to the squares of their diameters* (§§ [illegible]).

EXERCISE 104

1. Find the area of a sector in a circle of 8″ radius, the arc of the sector being 60°.*

(Is the area of a sector $= \frac{1}{2}$ arc $\cdot R$?)

2. Show that similar sectors are in the same ratio as the squares of their radii (*similar sectors* have the same number of degrees in their central angles).

3. Find the area of a sector whose arc is 45° and whose radius is 24″.

(The computation in examples 4 to 8 is all to be done mentally.)

4. The area of a circle is 49(3.1416). Find the diameter.

5. The area of a circle is $48\,\pi$. Find the radius.

6. The area of a circle is $98\,\pi$. Find the radius.

7. The area of a circle is $(3+2\sqrt{2})\pi$. Find the radius, correct to three decimal places.

8. The area of a circle is $(7+2\sqrt{12})\pi$. Find the area, correct to three decimal places. Find the radius.

9. The length of a chord is 15″. The arc the chord subtends is 60°. Find the area of the triangle formed by the chord and the radii drawn to its extremities.

10. The length of a chord is 15″. The arc the chord subtends is 38°. Find the area of the triangle formed by the chord and the radii drawn to its extremities.

11. The length of a chord is 18″, the arc it subtends is 60°. Find the area of the segment formed by the chord and the arc.

12. The arc of a segment is 42°; its chord is 41″. Find the area of the segment.

13. Find the radius of a circle inscribed in a triangle each side of which is 16″.

* A *sector* is the portion of a circle bounded by an arc and the radii drawn *to the extremities* of the arc.

14. Find the area of the circle circumscribed about the triangle in example 13.

15. The sides of a triangle are 10″, 24″, 26″. Find the areas of the circumscribed and inscribed circles.

16. The sides of a triangle are 15″, 20″, 25″. Find the areas of the inscribed and circumscribed circles.

17. Inscribe an octagon in a circle whose radius is 12″. Find the area of the octagon.

18. Find area of polygon of 16 sides inscribed in a circle of 12″ radius. Compare the areas of the polygons and circle of examples 17 and 18.

19. Find the perimeter of a pentagon inscribed in a circle of 12″ radius.

20. Find the perimeter of a decagon inscribed in a circle of 12″ radius.

21. The radii of two circles are in the ratio 1 to 4. Compare their areas.

22. The areas of two circles are in the ratio 1 to 4. Compare their radii.

23. The area of a circle is 64π. Find the radius of a circle four times as large.

24. The areas of two circles are in the ratio $\frac{9}{16}$. The radius of the first is 7″. Find the radius of the second.

25. The areas of two circles are in the ratio $\frac{9}{16}$. The circumference of the smaller is 81. What is the circumference of the larger?

26. Two similar sectors have an angle of 37° 20′. The arc of the smaller is 21′; the ratio of their areas is $\frac{49}{64}$. Find the arc of the larger.

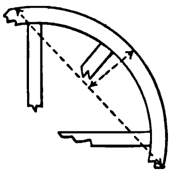
27. The arcs of two similar sectors are in the ratio $\frac{5}{6}$. An arc of the larger is $17\frac{1}{2}$″. The angle at the center is 31′ 15″. Find the arc of the smaller.

28. The area of one circle is double that of another. What is the ratio of their radii?

29. A cylinder is 10″ high, 6″ radius. Find the radius of a cylinder four times as large but having the same length.

30. The areas of two circles are 728 and 364, respectively. Find, mentally, the ratio of their radii, correct to three decimal places.

31. From the extremity of a 16″ tangent to a circle a 24″ secant is drawn. Find the segments into which the circumference divides the secant.



32. A wheelwright is given a part of a broken wheel to restore the wheel. To restore it he needs the diameter. He measures the chord of the arc given him, 24″; the height of the segment is 4″. How large was the wheel?

Loci

294. Locate a point in a plane which is equidistant from two intersecting lines. Such a point has an indefinite number of positions (§ 130), and since a line is the path of a moving point, this point describes a line (§ 55). Such a line is called the **locus of points** satisfying a given set of conditions (§ 131).

EXERCISE 105

1. Find the locus of points in a plane which are 3″ from a given fixed point in the plane.

2. In what position is the point which is equidistant from two given points?

3. Find the position of all the points 5″ from a given line.

4. What is the locus of points equidistant from two parallel lines?

5. Find the locus of points such that the distance of the point from the Y-axis is double its distance from the X-axis. Is the locus a straight line?

6. Find the locus of points such that the distance from the X-axis is double the distance from the Y-axis. Can you form the equation which satisfies this condition?

7. Draw a circle with a 10″ radius. In this circle draw 6 parallel chords and find the locus of their middle points.

8. AB is an 8″ line. Through A ten lines are drawn, and through B ten lines are drawn, respectively, perpendicular to the lines through A. Find the locus of the vertices of the right angles formed.

9. The radius of a circle is 6 inches, and through a point 12 inches from the center of the circle lines are drawn to meet the circumference. Find the locus of the middle points of these lines.

10. Through a point P on the circumference of a circle chords PA, PB, PC, etc., are drawn. Find the locus of the middle points of these chords.

11. The base of a triangle is 10″. The median drawn to the base is 8″. Find the locus of the vertex.

12. A number of straight lines intersect two parallel lines. Find the locus of their middle points.

13. The base and one side of a triangle are known. Find the locus of the vertex.

14. Find the locus of points equidistant from two adjacent sides of a quadrilateral; from the other two sides. Do these loci form one straight line? Do they ever form one straight line?

15. The center of a parallelogram is the intersection of the diagonals. Plot the locus of the centers of all parallelograms which have a fixed base and a side of fixed length.

16. Find the locus of the centers of all circles tangent to a given circle at a fixed point.

17. Find locus of the middle points of all equal chords of a given circle.

18. The diameter of a circle is 20″, an indefinite number of chords of 16-inch length are drawn. Find the area inclosed by the locus of the middle points of these chords.

19. Two chords pass through the same point. The segments of one chord are $x+6$ and $x-5$. The segments of the second chord are 6 and 7. Find the lengths of the segments of the first chord.

20. Two chords intersect. The segments of the first chord are $x+7$ and $x+2$. The segments of the second chord are 6 and 1. Find the length of the first chord.

21. Two chords intersect at right angles. The segments of the first are $x-8$ and $x-2$. The segments of the second are 4 and 4. Find diameter of circle.

22. Two chords intersect at right angles. The segments of the first are $x+10$ and $10-x$. The segments of the second are 8 and 8. Find radius of circle, and distance of second chord from center.

23. Two chords intersect. The segments of the first are $2x+7$ and $x-5$. The segments of the second are 3 and 23. Find the length of the first chord.

24. From a point without a circle a secant and a tangent are drawn. The tangent is 12 inches long. The segments of the secant are x and y. The difference between these segments is 28. Find the length of the secant.

25. From a point without a circle a secant and a tangent are drawn. The tangent is $9\sqrt{3}$ inches long. The segments of the secant are x and y. The difference between these segments is 9. Find the length of the secant.

Supplemental Applied Mathematics

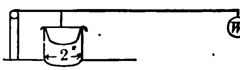
1. A steam pump delivers 2.35 gallons of water per stroke and runs 48 strokes per minute. How many gallons will it deliver per hour?

To find the size of a boiler feed pipe. Good practice allows a velocity of 300 feet per minute; a gallon of water equals 231 cu. in., If we let D equal the diameter of the pipe, then $\frac{\pi D^2}{4}$, or $.7854\, D^2$, equals the area. If the pipe carries water at a speed of 300 feet or (300×12) inches per minute, then the number of cubic inches passing any point per minute is $\frac{\pi D^2}{4} \cdot 3600$; to reduce cubic inches to gallons we divide by 231.

2. A battery of boilers requires 6000 gallons of water per hour. (*a*) Compute the size of feed water pipe required. (*b*) What size pipe should be ordered?

3. A 4″ feed water pipe is required to feed a battery of boilers. How many gallons an hour are used?

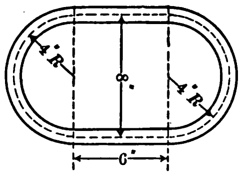
4. The pressure of air is 14.7 pounds per square inch. The air has been exhausted from a steel cylinder 5″ in diameter. Find the pressure on the end of the cylinder.



5. A safety valve is set to pop at 160 pounds. The short arm of the valve lever is 3″, the long arm 15″. The diameter of the valve is 2″. What weight must be attached to the lever?

6. In the above problem W is 25. What is the diameter of the valve, other values being the same as in problem 43?

7. The diameter of a safety valve is 2″. The length of the lever is 14″. The weight W is 30 pounds. At what point on the lever must the valve be attached if the maximum boiler pressure is 120 pounds?



8. Two 6″ steam mains are to lead into one header. Find diameter of header having an area equal to that of the two 6″ mains.

9. The steel link shown is made of $\frac{3}{4}''$ steel. Find the weight of it, considering it to be of circular cross section.

Power is often transmitted by belt gearing or cogwheel gearing. When two wheels are geared as in Figure A or Figure B, and one of these wheels is driven, the relative number of revolutions of the wheels is dependent on their circumferences. In Figures A and B it is evident that the smaller wheel revolves the faster. The law for the revolutions of such wheels may be stated:

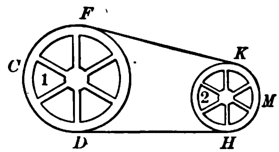


FIGURE A

The number of revolutions is inversely proportional to the diameters.

If the two wheels are equal, they will make the same number of revolutions.

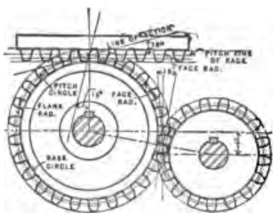


FIGURE B

If two equal pulleys are belted together, it is evident that the belt rests on one half the circumference of the pulley, that is, covers 180° of the circumference of the wheel. This part of the circumference touched by the belt is called the *arc of contact*. In Figure A, pulley 1, arc *FCD* is evidently greater than 180°; and in pulley 2, arc $HMK < 180°$.

The *surface speed* of a wheel is dependent upon the number of revolutions and the diameter of the wheel. That is, if a wheel is 10″ in diameter and makes 100 R. P. M. (revolutions per minute), the surface speed is $100(10\,\pi)'' = 1000\,\pi'' = 3141.6''$. Manufacturers of machines provide means for changing speeds of lathes, milling machines, boring mills, etc., within the limits of good practice.

For the speed of grindstones and emery wheels the following may be taken as representing good practice:

Grindstones { Machinist's tools, 800–1000 feet per minute.
Carpenter's tools, 550–600 feet per minute.

Grindstones for very rapid grinding $\left\{\begin{array}{l}\text{Coarse stones — Ohio, 2500 feet per minute.}\\ \text{Fine stones — Huron, 3000–3400 feet per minute.}\end{array}\right.$

Emery wheels 5500 feet per minute.

Sometimes the rule is given for grindstones as follows:

"Run at such speed that the water just begins to fly." This is a speed of about 830 feet per minute and would be a good average speed for a general purpose stone. A good ready rule for emery stones is to run them at a surface speed of a mile a minute.

An example or two will illustrate the method of calculating surface speeds.

How fast should a Huron stone, 3 feet in diameter, be run to give a surface speed of 3400 feet per minute?

Circumference of stone $= 3.1416 \times 3 = 9.42$.

For one revolution, then, a point on the circumference of the stone travels 9.42 feet. Hence, to travel 3400 feet per minute it must turn

$$\frac{3400}{9.42} = 373 \text{ revolutions (nearly).}$$

Be sure in getting the surface speed of any wheel to reduce *inches* to *feet*. For instance, in getting the number of revolutions of an 8″ wheel to give a surface speed of 4000 feet,

$$\frac{(3.1416)8}{12} = 2.095,$$

$$\frac{4000}{2.095} = 1910 \text{ revolutions.}$$

Cutting speeds on lathe and boring mill work may be calculated in the same way.

Circumference of work $\times$ revolutions per minute $=$ cutting speed.

$$\frac{\text{Cutting speed (in feet)}}{\text{Circumference of work (in feet)}} = \text{revolutions.}$$

In calculating the cutting speed of a drill take the speed of the drill circumference. For instance, a $\frac{1}{2}''$ drill is making 300 revolutions per minute. What is the cutting speed?

$3.1416 \times \frac{1}{2} = 1.57^+ =$ circumference in inches. $\frac{1.57}{12} =$ feet.

$$\frac{1.57}{12} \cdot 300 = 39.3 \text{ feet per minute, cutting speed.}$$

Tables of proper cutting speed are given in many handbooks as so many feet per minute. To find the necessary revolutions divide the cutting speed by the circumference of the work.

10. A 6″ shaft pulley makes 720 R. P. M. The machine it drives is to make 360 R. P. M. What is the diameter of the pulley on the machine?

11. Another pulley on the above shaft is 12″. To what size pulley must it be belted so that the machine it drives may make 1080 R. P. M.?

12. A lathe spindle is running 2000 R. P. M. What is the speed of a point on the surface of a 4″ block placed in the chuck?

13. A point on a 4″ block placed in the chuck of a lathe is traveling 1 mile per minute. How many R. P. M. must the lathe spindle make if a point on a 6″ block travels at the same rate? How many R. P. M. is the speed of the spindle decreased?

14. A shaft carrying a 16″ pulley runs 250 R. P. M. This pulley is belted to a lathe whose spindle must run 665 R. P. M. Find the diameter of the pulley on the headstock of the lathe.

Remember that pulleys only vary by half-inch sizes. Choose nearest half inch to size calculated.

15. A trolley car has 30″ wheels. The speed of the motor is 323 R. P. M. The cogwheel on the axle is 8″, on the motor 12″. How fast can the car travel?

16. A casting is 30″ in diameter. Find number of revolutions necessary for a cutting speed of 40 feet per minute.

17. Speed of line shaft 250. Size of pulley on wheel spindle 6″. Size of emery wheel 12″. What size of pulley should be used on the line shaft to give a surface speed of 5000 feet?

18. A 30″ pulley on the line shaft runs 200 revolutions. The emery wheel spindle has a 6″ pulley on it. What size of emery wheel should be used for a speed of 5500 feet?

19. A grindstone for carpenter's tools $3\frac{1}{2}$ feet in diameter is to be belted from a line shaft running 225 revolutions. Select

two pulleys of proper ratio for a surface speed suitable to this class of work.

20. The Bridgeport Safety Emery Wheel Co., Bridgeport, Conn., build an emery wheel 36″ in diameter and recommend a speed of 425–450 revolutions. Calculate the surface speed at 425 and 450.

21. A large surfacer runs at 3600 R. P. M. The two belt pulleys on the arbor are $4\frac{1}{2}''$ diameter. Find the belt travel in feet per minute.

22. Calculate the belt speed on a high-speed automatic engine carrying a 48″ pulley and running at 250 R. P. M.

Formula for length of an *open* belt to connect two pulleys of known diameter.

L = length of belt in inches.
D = diameter of large pulley.
d = diameter of small pulley.
C = distance between centers.

$$L = 3\tfrac{1}{4}\left(\frac{D+d}{2}\right) + 2\,C.$$

This formula in words reads:

$3\frac{1}{4}$ times the sum of the pulley diameters divided by 2, plus twice the distance between the centers. The formula for a *crossed* belt is very nearly the same.

$$L = 3\tfrac{3}{8}\left(\frac{D+d}{2}\right) + 2\,C.$$

Both these formulas give only the approximate length and cannot be relied on for actually cutting the belt.

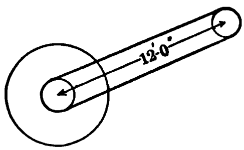
23. Find length of open belt to connect a 10″ and 16″ pulley 12 feet apart center to center.

(Solve by formula, then by geometry and trigonometry, and compute the amount of error made when the approximation formula was used.)

24. Solve problem 23 for a crossed belt, using the approximation formula.

25. Find number of revolutions per minute necessary to give a cutting speed of 40 feet per minute with a $\frac{1}{2}''$ drill.

26. A 1-inch drill, drilling tool steel, revolves at the rate of 62 R. P. M. Find the cutting speed.

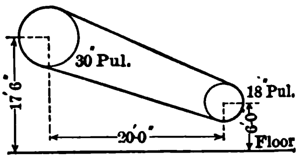


27. Stone $3\frac{1}{2}$ feet diameter for general purposes.

Speed of line shaft 150 R. P. M.

Select two pulleys to run it and compute length of belt.

28. Two pulleys are situated as shown. Find the length of belt to connect them.



29. The cars on the Aurora Chicago electric road are geared to 60 miles per hour. The car wheels controlled by the motor are 30" diameter. The axle cogwheel is 8". The cogwheel on the motor is 14". Find speed of motor when current is on full.

30. An engine flywheel 18 feet in diameter revolves at a speed of 90 R. P. M. Find the velocity in feet per minute of a point on the circumference.

31. The small sprocket of a bicycle contains 8 teeth, the large sprocket 24 teeth. The rear wheel is 22" in diameter. Find the distance traveled over by the bicycle for one revolution of the pedals.

32. A driving pulley makes 110 R. P. M., and is 21" in diameter. What should be the size of the driven pulley in order to make 385 R. P. M.?

33. The main shaft of a shop makes 120 R. P. M. A countershaft belted to it is to make 150 R. P. M. The following pulleys are on hand: 16", 24", 35", and 38". Can two of these be used or must a new pulley be ordered?

34. A driving pulley is 48" diameter and makes 90 R. P. M. The driven pulley is 20" diameter. How many revolutions *will it make?*

35. A pulley has six arms. What angle is included between the center lines of any two arms?

36. A pulley has eight arms. What angle is included between the center lines of any two arms?

37. A circle is 48″ in diameter. What is the length in inches of an arc of 1°?

38. A belt pulley 50.265 inches in circumference makes 150 R. P. M. (revolutions per minute). How many feet does the belt travel per minute?

39. A casting on a boring mill is 8.27 ft. in circumference. How many turns per minute should the table run to give a cutting speed of 45 ft. per minute on the outside of the casting?

40. A 14″ pulley is running a 26″ pulley. The 14″ pulley runs 225 revolutions per minute. What is the speed of the 26″ pulley?

41. A 12″ line shaft pulley runs 280 revolutions and is belted to a machine running 70 revolutions. What must be the size of the pulley on the machine?

42. The driving wheel of an engine $5\frac{1}{2}$ feet in diameter makes 4500 revolutions in going a certain distance. How many revolutions will an 8-foot wheel make in going the same distance?

43. A shop is run by a motor belted to a line shaft. The line shaft must run 300 R. P. M. The motor was to run at 1200 R. P. M. A 40″ pulley was put on the line shaft and a 10″ on the motor. On starting the motor was found to run 1400 R. P. M. Determine the proper size of pulley on the motor to give 300 R. P. M. at the line shaft.

44. Line shaft runs 250 R. P. M. Grinder with 6″ pulley is to run 1550 R. P. M. Determine size of pulley on line shaft to run grinder direct.

45. Speed of line shafting is 120 R. P. M. Diameter of pulley on line shafting is 15″. Speed of emery wheel required is 600. Find diameter of pulley on emery wheel.

Work. Whenever a force causes a body to move, work is done. Unless the body is moved, no work is done. A man may push against a heavy casting for hours, and unless he moves it no work is done, no matter how tired he may feel at the end of the time.

Unit of Work. The unit by which work is measured is called the foot pound. This is the work done in overcoming the resistance of one pound through a distance of one foot; that is, if a weight of one pound is lifted one foot, work equal to one foot pound is done. All work is measured by this standard. The work in foot pounds is the product of the force in pounds and the distance in feet.

If a weight of 80 pounds is lifted a distance of four feet, $80 \times 4'$ or 320 foot pounds of work are done. It would require the same amount of work to lift 40 pounds 8 feet or 20 pounds 16 feet. The force of steam pressing against a piston does work when it moves the piston against a resistance. The work accomplished may be found by multiplying the total force acting on the piston in pounds by the distance through which it moves, measured in feet. The total force on a piston is, of course, the area of the piston in square inches multiplied by the pressure in pounds per square inch.

Illustrations:

The work necessary to pump a certain amount of water is the weight of the water × the height through which it is lifted or pumped. The work necessary to hoist a casting is the weight of the casting × the height to which it is lifted. The work done by a belt is the effective pull of the belt × the distance which the belt travels in feet. The work done in hoisting an elevator is the weight of cage and load × the height of the lift. Numerous other illustrations of work will suggest themselves to the student.

Power. Power is the rate of doing work. That is, in calculating power, the time required to do a certain number of foot pounds of work is considered. If 10,000 pounds are lifted 7 feet, the work done is 70,000 foot pounds. If one machine can do this in half the time another machine takes, the first machine has twice the power. The work done by each machine, however, is the same. The engineer's standard of power is the horse power, which may be defined as the ability to do 33,000 *foot pounds of* work per minute. If a weight of 6600 pounds is to be

raised 10 feet, it will require $6600 \times 10 = 66,000$ foot pounds. If this is done, in two minutes 33,000 foot pounds must be done in one minute, or the work required may be done by expending one horse power for two minutes. The horse power required to perform a certain amount of work is found by dividing the foot pounds done per minute by 33,000.

46. A crane lifts a casting weighing 3 tons, 20 feet from the floor in 30 seconds. What is the horse power used?

3 tons = 6000 pounds.

$6000 \times 20 = 120,000$ foot pounds done.

120,000 foot pounds done in 30 seconds = 240,000 done in one minute.

$\frac{240000}{33000} = 7.27$ horse power used.

47. A casting weighs 300 pounds. How much work is required to place it on a planer bed 3 feet 5 inches above the floor?

48. How much work is required to pump 5000 gallons of water into a tank 150 feet above the pump?

49. The average pressure on a steam piston is 60 pounds per square inch. The diameter of the piston is 32″, the stroke is 54″. Find the work done per stroke. What is the work per revolution?

50. If the engine in example 49 makes 94 revolutions per minute, how much work is done per minute?

51. 33,000 foot pounds per minute = one horse power. What is the horse power of the engine mentioned above?

52. A belt is carried by a 36″ pulley running at 150 R. P. M. The effective pull in the belt is 240 lb. Find the horse power.

53. A pumping engine lifts 92,500 gallons of water every hour to a height of 150 feet. What is the horse power of the engine?

CHAPTER XVIII

Application of Exponents

295. In Chapter XIII the fractional exponent was treated, but most of the exponents were in the form of common fractions. We shall now extend the study to cover decimal fractions.

When the power indicated can be found by means of factoring no difficulty arises. Often, however, one must resort to tables already prepared just as was necessary in the trigonometric functions other than of 30°, 45°, 60°. (§ 198.)

The same rules for exponents will hold as were used in Chapter XIII; namely, *in multiplication, add the exponents of like numbers; in division, subtract the exponents in the divisor from the exponents of like numbers in the dividend: to find the root of a number, divide its exponent by the index of the root; this gives the exponent of the root: in involution, multiply the exponent by the index of the power to which the number is to be raised.*

Ex. Multiply 5^x by 5^y.

$$5^x \cdot 5^y = 5^{x+y}.$$

EXERCISE 106

Perform the following indicated operations:

1. $10^{.25}(10^{.75})$.
2. $10^{.333+}(10^{.6666+})$.
3. $(10^{.25})^4$.
4. $(10^{.2})^5$.
5. $(81^{.25})^4$.
6. $81^{\frac{1}{4}}$.
7. $216^{.333\frac{1}{3}}$.
8. $\dfrac{(10^{.3010})^2(10^{.6990})^2}{10^2}$.
9. $\dfrac{(10^{.4771})(10^{.5229})}{10^2}$.
10. $16^{.25}(16^{.5})$.
11. $(256^{.125})(32^{.2})$.
12. $10 \div 10^{.3010}$.
13. $25^{.8263} \div 25^{.3263}$.
14. $1024^{.31} \div 1024^{.11}$.

Find approximate results:

15. $28^{.33\frac{1}{3}}$.

Since $27^{.33\frac{1}{3}} = 3,\ 28^{.33\frac{1}{3}} = 3+$.

16. $10^{.5}$. **17.** $10^{.56}$. **18.** $10^{.3333}$. **19.** $10^{.3010}$.

Is the result in example 19 greater or less than the result in example 18? Will any of the results in examples 16–19 produce a two-place integral number? Or a two-place integral number plus a decimal?

20. $10^{.4771}$. **21.** $10^{.6990}$.

22. Between what two numbers, not fractions, must the exponent lie to raise 10 to such a power as will produce 2? 3? 4? 5? 6? $10^0 = ?$ $10^1 = ?$ Is the exponent in the result of example 22 a proper or an improper fraction? That is, is it a decimal, or is it an integer plus a decimal?

23. $10^1 = ?$ $10^2 = ?$ Between what integers does the exponent lie to raise 10 to such a power as will produce 25? 30? 75? Is this exponent a mixed decimal; that is, an integer plus a decimal?

296. The results of exercise 106 show that it is possible to raise a number to such a power as will produce any required number. The number which is raised to a power is called the **base**. The base which is used for all practical purposes is 10 and is the base used in the **common system** of exponents. With only a few of these exponents known it is possible by means of factors to find many other exponents.

Ex. 1. Given $10^{.3010} = 2$, to find the exponent which produces 5.

$$5 = \tfrac{10}{2},$$
$$10^1 = 10,$$
$$10^{.3010} = 2.$$

Hence, $10 \div 2 = 10^1 \div 10^{.3010} = 10^{1-.3010} = 10^{.6990}$.

That is, 10 raised to the power .6990 is 5.

Ex. 2. Given $10^{.4771} = 3$, to find the exponent which will produce 6.

$$6 = 2 \cdot 3, \qquad 10^{.4771} = 3, \qquad 10^{.3010} = 2.$$

Hence, $$10^{.4771}(10^{.3010}) = 10^{.7781} = 6.$$

Such exponents of 10 as we have been using are called **logarithms.**

EXERCISE 107

Given $\log 2 = .3010$; $\log 3 = .4771$; $\log 7 = .8451$; to find the logarithms of the following:

1. 21.

$$\begin{aligned} 21 &= 3 \cdot 7, \\ \log 3 &= .4771 \\ \log 7 &= \underline{.8451} \\ \log 21 &= 1.3222 \end{aligned}$$

Did you expect the logarithm of 21 to lie between 1 and 2? Why?

2. 42.	**7.** 98.	**12.** $\frac{27}{2}$.	**17.** $24 \cdot 9$.
3. 25.	**8.** 125.	**13.** $\frac{32}{3}$.	**18.** $63 \cdot 21$
4. 84.	**9.** 120.	**14.** $11\frac{1}{9}$.	**19.** $17\frac{6}{7}$.
5. 48.	**10.** 144.	**15.** 245.	**20.** $9\frac{18}{25}$.
6. 63.	**11.** $\frac{25}{2}$.	**16.** 1024.	

21. $\sqrt{\frac{5}{2}}$.

$$\sqrt{\frac{5}{2}} = \left(\frac{5}{2}\right)^{\frac{1}{2}} = \frac{5^{\frac{1}{2}}}{2^{\frac{1}{2}}}.$$

$$\begin{aligned} \log \frac{5^{\frac{1}{2}}}{2^{\frac{1}{2}}} &= \log 5^{\frac{1}{2}} - \log 2^{\frac{1}{2}} \\ &= \tfrac{1}{2}\log 5 - \tfrac{1}{2}\log 2 \\ &= \tfrac{1}{2}(.6990) - \tfrac{1}{2}(.3010) \\ &= \tfrac{1}{2}(.6990 - .3010) \\ &= \tfrac{1}{2}(.3980) \\ &= .1990. \end{aligned}$$

22. $\sqrt[3]{14}$.

23. $\sqrt[5]{14}$.

24. $\dfrac{3^{\frac{1}{4}}}{2^{\frac{1}{2}}}$.

25. $\sqrt[8]{32}$.

26. $30^{\frac{2}{3}} \cdot 25^{\frac{1}{3}}$.

27. $\sqrt[9]{35}$.

28. $\dfrac{7^{\frac{1}{3}}}{5^{\frac{1}{4}}}$.

29. $2^{\frac{5}{8}}$.

30. $\sqrt[6]{125}$.

31. $(84)^{\frac{1}{3}} \div (21)^{\frac{1}{4}}$.

32. $\frac{8^{\frac{2}{5}}}{2^{6}}$.

297. A **logarithm** is made up of two parts, a **characteristic** and a **mantissa.** The decimal part of a logarithm is the *mantissa,* the integral part is the *characteristic.*

In the common system of logarithms, the mantissæ of all numbers having the same sequence of figures is the same, regardless of the position of the decimal point.

This is readily seen when one remembers that our number system is a decimal one, and the base of the common system is 10. *E.g.* find logarithms of 24, 240, 2400.

$$\log 24 = \log 2^3 \cdot 3 = 3 \log 2 + \log 3 = .9030 + .4771 = 1.3801.$$
$$\log 240 = \log 10 \cdot 24 = \log 10 + \log 24 = 1 + 1.3801 = 2.3801.$$
$$\log 2400 = \log 100 \cdot 24 = \log 100 + \log 24 = 2 + 1.3801 = 3.3801.$$

In a set of tables it is not necessary to print the characteristic. The mantissa is given in the table, and the characteristic is determined by the position of the decimal point.

Any number between 1 *and* 10 *has a mantissa only for logarithm.*

Any number between 10 *and* 100 *has* 1 + *mantissa for logarithm.*

Any number between 100 *and* 1000 *has* 2 + *mantissa for logarithm.*

Any number between 1000 *and* 10000 *has* 3 + *mantissa for logarithm.*

That is, the *characteristic* is *one less* than the number of digits in the integral part of the number.

$.1 = \frac{1}{10} = 10^{-1}$. That is, .1 has -1 for its logarithm.

$.01 = \dfrac{1}{100} = \dfrac{1}{10^2} = 10^{-2}$. That is, .01 has -2 for its logarithm.

$\log .001 = -3$, etc.

Those numbers between $\frac{1}{10}$ and $\frac{1}{100}$, or .1 and .01, must have logarithms which lie between -1 and -2.

The logarithms of numbers between .01 and .001 lie between -2 and -3.

Since the mantissæ for the same sequence of figures do not change, we have only to distinguish the logarithms of decimal fractions by their characteristics.

Thus,
$$\log 2.4 = .3801.$$
$$\log .24 = -1 + .3801, \text{ or } 9.3801 - 10,$$
$$\log .024 = -2 + .3801, \text{ or } 8.3801 - 10,$$
$$\log .0024 = -3 + .3801, \text{ or } 7.3801 - 10.$$

Note that .24 lies between 1 and .1 and its logarithm is -1 plus a decimal. *A convenient method of determining the characteristic of a decimal is to note the number of columns the first significant figure of the decimal is removed from the units' column.* For example, in .24 the 2 is one place from units'. The characteristic is -1. In .024, the 2 is two places removed and the characteristic is -2. Since it is not convenient to add positive decimals, or mantissæ, to negative characteristics, the logarithms of decimal fractions are written:

$$9. + \text{decimal} - 10,$$
$$8. + \text{decimal} - 10,$$
$$7. + \text{decimal} - 10, \text{ etc.}$$

EXERCISE 108

Find logarithms of the following:

1. .375.

The characteristic is $9 - 10$.

$$375 = 5^3 \cdot 3.$$
$$\log 375 = 3 \log 5 + \log 3$$
$$= 2.0970 + .4771$$
$$= 2.5741.$$

The mantissa of log .375 is therefore .5741.

Then $\log .375 = 9.5741 - 10.$

2. $.063 \cdot 210$.

$$
\begin{aligned}
63 = 3^2 \cdot 7;\ \log 63 &= 2 \log 3 + \log 7 \\
&= .9542 + .8451 \\
&= 1.7993. \\
\log .063 &= 8.7993 - 10. \\
210 = 2 \cdot 5 \cdot 3 \cdot 7;\ \log 210 &= \log 2 + \log 3 + \log 5 + \log 7 \\
&= .3010 + .4771 + .6990 + .8451 \\
&= 2.3222. \\
\log .063 \cdot 210 &= \log .063 + \log 210 \\
&= 8.7993 - 10 + 2.3222 \\
&= 11.1215 - 10 \\
&= 1.1215.
\end{aligned}
$$

The written work should stand as follows:

$$
\begin{aligned}
\log .063 &= 8.7993 - 10 \\
\log 210 &= \underline{2.3222} \\
\log .063 \cdot 210 &= 11.1215 - 10 \\
&= 1.1215.
\end{aligned}
$$

3. $.125$.
4. $.0225$.
5. $.1024 \cdot (10.24)$.
6. $.24(.09)$.
7. $1.44(.0144)$.
8. $.056$.
9. $.216$.
10. $\sqrt[3]{8.4}$.
11. $\sqrt{1.215}$.
12. $\sqrt[4]{17.15}$.
13. $(.027)^3$.
14. $(.0343)^2$.
15. $(2.25)^{\frac{1}{3}}$.
16. $.00324$.
17. $(.00081)^3$.
18. $.00032$.

THE TABLE*

298. The tables on pages 1 to 20 give the mantissæ of logarithms of all numbers from 1 to 10009, calculated to five-place decimals.

To find the logarithm of a number of four figures, look in the column headed N for the first three significant figures of the number. The mantissa of the logarithm is in the horizontal row with the first three figures of the number and in the column headed by the fourth figure of the number. To this mantissa prefix the required characteristic.

* Bowser's Tables, pp. 1–82.

Ex. 1. $\log 4567 = ?$

This is found by looking in the number column for 456, then across in a horizontal line to a column headed 7.

Ex. 2. Find log .03245.

The characteristic is $8. - 10$.

The mantissa is found by looking in the row with the number 324 and in the column headed 5.

For dividing a number having a negative characteristic, it is necessary to so arrange the characteristic that the negative part may be divided by the divisor with 10 for a quotient. Thus, if a characteristic is $9. - 10$ and the logarithm of the cube root is to be found, the characteristic is written $29. - 30$, whence

$$\begin{array}{r} 3\overline{)29. - 30} \\ 9. - 10 \end{array}$$

Ex. 3. Find $\log (.03257)^{\frac{1}{4}}$.

$$\begin{aligned} \log (.03257)^{\frac{1}{4}} &= \tfrac{1}{4} \log .03257. \\ \log .03257 &= 8.55267 - 10. \\ \tfrac{1}{4} \log .03257 &= \tfrac{1}{4}(8.55267 - 10) \\ &= \tfrac{1}{4}(38.55267 - 40) \\ &= (9.638167 - 10). \end{aligned}$$

In finding the logarithm of a number containing more than four figures it is necessary to correct the mantissa found in the table.

For example, $\log 4324 = 3.63589.$
$\log 4325 = 3.63599.$

Log 4324.5 must lie halfway between the logarithms of 4324 and 4325. The difference between the logarithms is 10. One half or .5 of this difference is 5. Hence,

$$\begin{aligned} \log 4324.5 &= 3.63589 + .00005 = 3.63594. \\ \log 4324.7 &= \log 4324 + .7 \text{ of the difference.*} \\ \log 43.242 &= \log 43.24 + .2, \text{ the tabular difference.} \end{aligned}$$

That is, the figures of the number to the right of the fourth figure must be considered decimal in making a correction.

* This difference between logarithms of two consecutive numbers is called *the tabular difference.*

Ex. 4. Find log .785324.

$$\log .7853 = 9.89504 - 10.$$
$$\log .7854 = 9.89509 - 10.$$

Tabular difference is 5, that is, .00005.

$$\begin{aligned}\log .785324 &= \log .7853 + .24 \text{ of } 5\\ &= 9.89504 - 10 + .000012\\ &= 9.895052 - 10.\end{aligned}$$

EXERCISE 109

Find the logarithms of the following:

1. 23496.
2. 318257.
3. .24968.
4. .032581.
5. .0674329.
6. 74.8521.
7. 8496.25.
8. 319.575.
9. .30105.
10. 84.6326.

The table of proportional parts saves multiplication if the logarithm of a five-place number is required.

11. 2.4976.

log 2.497 = .39742.

The tabular difference is 17. Looking in the proportional parts (P. P.) column under 17, we find the correction for 6 to be 10.2. This same correction would have been obtained had we multiplied 17 by .6.

12. $(14.698)(2.3476)$.
13. $(.29768)^2$.
14. $(3.7989)^2$.
15. $(41.867)^{\frac{1}{2}}$.

299. In finding the number corresponding to a logarithm we must reverse the operations of § 298.

Ex. 1. Find by means of logarithms (28.34)(.3265).

$$\log (28.34)(.3265) = \log 28.34 + \log .3265.$$
$$\begin{aligned}\log 28.34 &= 1.45240.\\ \log .3265 &= \underline{9.51688 - 10.}\\ \log (28.34)(.3265) &= 10.96928 - 10\\ &= .96928.\end{aligned}$$

Look in the table for the next lower *mantissa* to 96928. This is found to be (page 18) exactly 96928, so no correction is necessary. The number corresponding to this logarithm is 9317, and since our characteristic is 0, the required product of 28.34 and .3265 is 9.317.

Ex. 2. Find $6.8746 \div 24.79$.

$$\begin{aligned}
\log(6.8746 \div 24.79) &= \log 6.8746 - \log 24.79.\\
\log 6.8746 &= .837246.\\
\log 24.79 &= \underline{1.39428.}\\
\log(6.8746 \div 24.79) &= 9.442966 - 10\\
&= 9.44297 - 10.
\end{aligned}$$

(Before making this subtraction we must add $10 - 10$ to the .837246.) Looking in the table for the next lower mantissa, we find 44295 on page 5, the logarithm of 2773.

Our logarithm is 2 more than 44295. The tabular difference is 16. Our required result then is $2773\frac{2}{16}$, or 2773125. Our characteristic is $9 - 10$ or -1. That is, the first figure of the result is one place to the right of units' column (§ 297, page 324). Our quotient is therefore .2773125.

Ex. 3. The hypotenuse of a right triangle is 2973.5. One leg is 2279.6. Find the other leg.

Let $\qquad x =$ the required leg.

Then $\qquad x^2 = (2973.5)^2 - (2279.6)^2.$

$$x = \sqrt{(2973.5)^2 - (2279.6)^2}.$$

Since we cannot subtract by means of logarithms, this expression must be reduced to a factor form.

(Use type I.)

$$\begin{aligned}
x &= \sqrt{(2973.5 + 2279.6)(2973.5 - 2279.6)}.\\
&= \sqrt{(5253.1)(693.9)}.\\
\log x &= \tfrac{1}{2}\log 5253.1 + \tfrac{1}{2}\log 693.9.
\end{aligned}$$

The logarithm may now be found and the value of x found. This saves the squaring and the extraction of roots of large numbers.

EXERCISE 110

Find by logarithms the values of the following:

1. $\dfrac{(2.763)(48.21)}{1.684}$.
2. $\dfrac{(.2438)(169.65)}{(2.983)(31.84)}$.
3. $\dfrac{(3.015)^2}{\sqrt[3]{17.5}}$.
4. $3.1416(247)^2$.
5. $\dfrac{(47.6)^{\frac{1}{2}}(34.6)^{\frac{1}{3}}}{\sqrt[4]{2}}$.

6. The radius of a circle is 24.5″. Find area, correct to four decimal places.

7. The radius of a circle is $24\frac{3}{16}$″. Find the circumference. Find the area.

8. The diameter of a circle is $14\frac{3}{4}$″. Find the area.

9. The area of a circle is 628.32. Find the radius.

10. The area of a circle is 314.16. Find the radius.

11. The area of a circle is 274.31. Find the radius.

12. One acute angle of an isosceles trapezoid is 45°. The upper base is 8.1 and the altitude is 6.325. Find the area.

13. One acute angle of an isosceles trapezoid is 30°. The area is 81.939, and the altitude is 6.325. Find the bases.

14. If a body falls from rest from any point above the earth's surface, the distance it falls is expressed by the equation

$$S = \tfrac{1}{2}gt^2,$$

where S = distance in feet, g the velocity acquired in one second, due to the force of gravity, and t the time in seconds. g has been found by experiment to be 32.15 feet or 980 centimeters.

How long will it take a body to fall 584′?

15. How long will it take a body to fall one fourth of a mile?

16. When a projectile is shot upwards, its velocity is

$$v = \sqrt{2gs}.$$

A ball is thrown upward 100′. Find the velocity with which it is thrown. How long will it take it to fall back to the earth?

17. A bullet was shot upward with sufficient velocity to reach an airship 3000 feet above the earth. With what velocity was it fired?

18. The time it takes a pendulum to make one vibration is determined by the equation $t = \pi\sqrt{\frac{l}{g}}$, where t is in seconds, and l is the length of the pendulum. In using the formula g and l must be in the same denomination; that is, if l is in centimeters, g must also be in centimeters.

The length of a pendulum is 6.512′. Find the time of one vibration.

19. Find the length of a pendulum which beats seconds.

20. Find the length of a pendulum which beats half seconds. Find the ratio of the lengths of the pendulums in examples 19 and 20.

300. Much of the labor involved in solving triangles by trigonometry is avoided by the use of logarithms.

Ex. Given right $\triangle ABC$, right-angled at B, with $\angle A = 37° \, 30'$, side $c = 24.85$; to find a and b.

$$\tan A = \frac{a}{c}.$$

Then, $\tan 37° \, 30' = \frac{a}{24.85}$; [$\tan 37° \, 30' = .76732$. (Page 202.)]

or,
$$\begin{aligned} a &= 24.85\,(\tan 37° \, 30') \\ &= 24.85\,(.76732). \\ \log a &= \log 24.85 + \log .76732. \\ \log 24.85 &= 1.39533. \\ \log .76732 &= \underline{9.88498 - 30.} \\ \log a &= 11.28031 - 10. \\ &= 1.28031. \end{aligned}$$

Hence, $a = 19.068.$

$$\cos A = \frac{c}{b},\ b = \frac{c}{\cos A};\ \cos A = \cos 37° \, 30' = .79335.$$

$$\begin{aligned} \log b &= \log c - \log \cos A \\ &= \log 24.85 - \log .79335. \\ \log 24.85 &= 1.39533. \\ \log .79335 &= \underline{9.89947 - 10.} \\ \log b &= 1.49586 \quad \text{(Ex. 2, § 299.)} \\ b &= 31.323. \end{aligned}$$

A good check for the solution of a triangle is to use the *parts found* to find a *given part*.

Thus in $\sin A = \frac{a}{b}$ use a and b to find A.

$$\log \sin A = \log a - \log b.$$
$$\log a = 1.28031.$$
$$\log b = \underline{1.49586.}$$
$$\log \sin A = 9.78445 - 10$$
$$\sin A = .60877.$$
$$A = 37^\circ\ 30\tfrac{1}{1375}'.$$

This is within $\frac{1}{1375}'$ of being correct.

The use of the logarithm of .76732 avoided finding the product of 24.85 and .76732. Since there is some labor in looking up the natural tangent of 37° 30′ and the logarithm of that decimal, mathematicians have still further aided computation by making tables of the logarithms of all the trigonometric ratios, so that instead of looking up the trigonometric ratio and then its logarithm, the logarithm may be found at once. For example, page 75 contains all the logarithms of sines, tangents, cotangents, cosines of 37° to 38°; also, the logarithms of the same trigonometric functions from 52° to 53°. Looking in the column headed L. Tang. in the same horizontal row with 30′, we find 9.88498. This is the same number we found when we looked up the logarithm of .76732. Likewise, looking for the logarithmic cosine of 37° 30′ (page 75), we find 9.89947. Similarly in the check, looking in the logarithmic sines for 9.78445, we find (page 75) that it is log sin 37° 30′. Since it would be difficult to obtain the characteristic from the angle itself, the *logarithm tables of trigonometric functions* have all of the logarithm expressed except the -10. This must be supplied in logarithms of all sines and cosines, all tangents less than 45° and all cotangents over 45°. (The cotangent is the reciprocal of the tangent.)

EXERCISE 111

Solve the following right triangles: ($\angle B$ is the right angle.)

1. $a = 2416$, $c = 3125$.
2. $a = 41.165$, $A = 32° \ 20''$.
3. $a = 3.145$, $b = 4.762$.
4. $b = 5.761$, $c = 4.521$.
5. $b = .2469$, $A = 24° \ 35'$.
6. $b = .0298$, $C = 38° \ 17'$.
7. $c = .3116$, $A = 24° \ 45'$.
8. $c = .09575$, $C = 52° \ 28'$.
9. $a = .2496$, $c = .2496$.
10. $a = 60$, $c = 10.392$.
11. $a = 93.528$, $c = 54$.

12. The side of a regular pentagon is 23.46. Find the area.

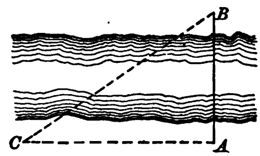
13. A regular pentagon is inscribed in a circle whose radius is 9.43′. Find the sides and area of the pentagon.

14. A statue stands on a pedestal and the relative heights of the two are such that to a person 66 feet from the foot of the pedestal the angle of elevation is 30°, while that of the top of the statue is 60°. Find the height of the statue.

15. From the top of a light tower 100 feet high, the angles of depression of two objects on a level with the foot of the tower and in line with the tower are 51° 32′ and 37° 28′, respectively. Find the distance the objects are apart.

16. A pyramid has a square base $ABCD$. The vertex V of the pyramid is directly over the center of the base; that is, the altitude VE meets the base at E, the intersection of the diagonals of the base. The faces VAB, VBC, VCD, VDA, of the pyramid are inclined 30° to the base; one side of the base is 20′. Find the slope of an edge VA. $\left(\text{This slope is } \dfrac{VE}{AE}.\right)$

17. A man at A wishes to know the distance of an inaccessible object B. To do this he measures $AC = 100$ yards, in a direction at right angles to AB. He then finds that the angle ACB is 39° 26′. What is the distance AB?



18. A ship several miles out is directly opposite the point where you are standing on the lake shore. Give method for determining the distance from the shore to the ship.

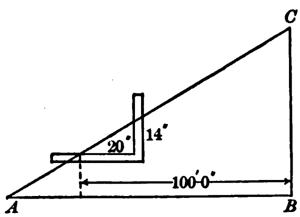
19. One base angle of an isosceles triangle is 37° 5′. One leg is 38.75′. Solve the triangle.

20. A man standing on a river bank notes that the angle of elevation of the top of a tree on the opposite bank is 21° 3′. 200 feet away from the bank he finds the angle of elevation of the top of the same tree to be 14° 10′. How wide is the river?

21. Walking along a straight road a traveler noticed at one milestone that a house was 30° off to the right. At the next milestone the house was 45° off to the right. How far was the house from the road? Is there more than one solution?

22. What is the length of guy wires necessary to support a smoke stack 175′ high, the wires being fastened to a ring around the stack 15 feet from the top, and making an angle of 27° with the stack?

23. The above stack is in the center of a lot 200′ by 100′. Find maximum angle guy wires can make with the stack and yet keep inside of property limits.



24. A man wishing to know the height of a smoke stack measures off 100 feet from its base. He then holds a square with the long arm horizontal and the short arm vertical, and sights the top C. Find $\angle A$ and the height of the stack if the observer's eye is 5′ from the ground.

25. A circle is inscribed in a right triangle. Show that the diameter of the circle is equal to the sum of the legs of the triangle minus the hypotenuse.

26. A circle is inscribed in a right triangle, a leg and the adjacent angle of which are 18′ and 42°, respectively. Find the diameter of the circle.

27. A circle is inscribed in a regular polygon whose area is 64 square inches and whose perimeter is 16 inches. Find the radius of the circle. Is your result reasonable? Find the area of the circle. Make a general statement covering the limits in such a case.

28. A circle, regular hexagon, and equilateral triangle all have the same perimeter. Compare their areas.

29. Find the altitude of the sun when a tower 83.5′ high casts a shadow 59.75′ long.

30. The radius of the earth is about 4000 miles. A certain lighthouse can be seen $16\frac{1}{2}$ miles out at sea. Find the height of the lighthouse.

31. The deck of a ship is 38′ above the water line. How far distant is the horizon of a person whose eyes are 5′ above the deck?

(A rule giving approximate results for distances at sea is

$$\sqrt{\tfrac{3}{2}\text{ the height in feet}} = \text{the distance in miles.}$$

Compare your result with that secured by this formula. This formula depends upon the fact that the curvature of the earth is 8″, or $\frac{2}{3}$′, to the mile.)

32. The light of a lighthouse is 75′ above the water. (*a*) How far can it be seen at sea when the observer is at sea level? (*b*) When the observer is on deck 30′ above the water line?

33. The radius of a circle is 132′. Find the length of an arc of 57.2958°.

34. A man in an aeroplane is flying in line with a horizontal road and vertically over the road. Two consecutive milestones have angles of depression of 47° and 32°. How far is the machine from the earth?

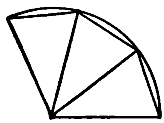
35. A switch-back railroad is at an angle of 7° 10′. How many feet to the mile does it rise?

36. In a circle whose radius is 15′ a chord 9′ long subtends a central angle of a certain number of degrees. How long a chord subtends an angle twice as great?

37. When the sun has an altitude of 60°, in what position must a 12-foot stick be held, one end resting on the ground, to cast the longest shadow? What is the length of the shadow?

38. The diagonals of a rhombus are 57.25″ and 39.75″. Find the sides and angles.

39. In turning a corner, the engineer of a street railway company laid out chords 75′ long. The angle between successive chords was 150°. Find the radius of the curve.



40. For over 100 miles east of Pike's Peak the plateau is nearly level. The peak rises 8800′ above the plain. From what distance east is it visible?

41. Mount Rainier, a beautiful volcanic mountain in Washington, is 14,400′ above the sea. With a telescope how far at sea can this mountain be seen?

42. The sides of a triangle are 10, 24, 26. Find the radii of the circumscribed and inscribed circles. Do the circles have the same center?

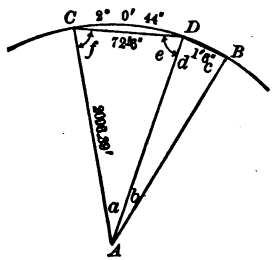
43. The sides of a triangle are 6, 8, 10. Find the area of the inscribed circle.

44. The sides of a triangle are 10, $7\frac{1}{2}$, $12\frac{1}{2}$. Find the radius of the inscribed circle.

45. One side of an equilateral triangle is 30. Find the radius of the inscribed circle.

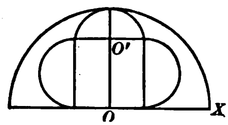
46. How large a circle can you cut from a triangular piece of tin whose edges are 25, 60, 65 centimeters, respectively?

47. A chord passing through a point within a circumference is divided by the point into segments 6 and 20. Another chord through this point has segments x and y. The sum of segments x and y is 23. Find x and y.



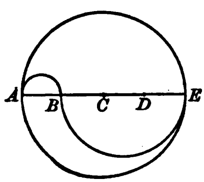
48. Given a $2° 44''$ curve on a radius of $2096.39'$, also chord $CD = 72'\text{-}6''$, chord $DB = 1'\text{-}6''$. Find angles a, b, c, d, e.

49. The rectangular part of a cathedral window is surmounted by a semicircle whose radius is $20''$. The semicircle incloses three equal semicircles, one tangent to the outer semicircle at the highest point of the window, the diameters of the three equal semicircles forming three sides of a square whose base coincides with the diameter of the outer semicircle. Draw the design to a scale.



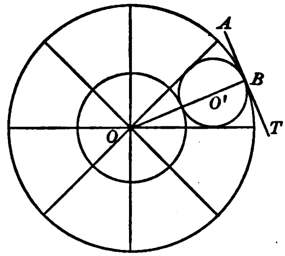
50. A party took a $28\frac{1}{2}$ mile drive in a machine in $2\frac{1}{4}$ hours. Part of the trip was over boulevards where the speed limit is 8 miles an hour; the rest was over roads allowing 15 miles an hour. If the driver ran his machine up to the legal speed limit, how much of the trip was over a boulevard?

51. A man drove a machine 50 miles in a given time. Had he not exceeded the speed limit it would have taken him 1 hour 15 minutes longer to make the trip. His rate exceeded the legal speed limit 9 miles per hour. What was the speed limit?



52. If a diameter AE is divided into four equal parts by the points B, C, D, and semicircles on AB, BE; AC, CE;

AD, *DE* as diameters are drawn, one above and one below *AE* in pairs, show that the four curvilinear areas formed are equivalent.



53. The central portion of a design for a leaded glass door consists of two concentric circles, the radius of the greater being 16″. In the circular band between these circumferences are eight equal circles, each tangent to two of the eight equal circles and to the concentric circles. Draw the design to a scale and find the radius of the inner concentric circle.

Supplemental Applied Mathematics for Girls

Problems on Quantity and Cost of Material

1. If it takes $3\frac{1}{2}$ cupfuls of flour to make one loaf of bread, how many quarts of flour will it take to make 5 loaves of bread?

SOLUTION: Let x = number of cupfuls of flour in 5 loaves.

By the conditions, $\frac{3\frac{1}{2}}{x} = \frac{1}{5}$.

Whence, $x = 17\frac{1}{2}$ cupfuls,

or, since 4 cupfuls = 1 quart,

there are $4\frac{3}{8}$ quarts of flour in 5 loaves.

2. From $\frac{1}{8}$ of a cupful of flour, 3 muffins can be made. How much flour will it take to make 1 dozen muffins?

3. From one loaf of bread 15 slices can be cut. How many loaves of bread will be required for a family of six for a day, if each one uses 2 slices per meal?

4. One loaf weighed $15\frac{1}{2}$ ounces. The crusts were removed and weighed $4\frac{1}{2}$ ounces. What per cent of the loaf was crust?

5. If the bread cost 5 cents per loaf, what was the cost of the crust? If a family used 3 loaves per week for toast from which the crusts were removed, how much would the waste amount to in a month, if the crusts were not utilized for other purposes? What per cent of the cost of the bread was waste?

6. How many potatoes are cooked in your home per day? What part of a bushel is this number if there are 144 medium-sized potatoes in a bushel?

7. How many bushels of potatoes are used in your home per month? When potatoes sell for 80 cents per bushel, how much is spent for potatoes in your home during a month?

8. When butter is 37 cents per pound (2 cupfuls) and butterine 25 cents per pound, how much will be saved by using $\frac{3}{4}$ of a cupful of butterine instead of butter, for a cake?

9. When eggs are 40 cents per dozen, and butter 35 cents per pound, which combination would be cheaper to use, $\frac{1}{2}$ cupful of butter and one egg, or $\frac{1}{3}$ cupful of butter and 2 eggs?

10. At the above prices for eggs and butter, which is cheaper to bake, a sponge cake requiring 6 eggs or a butter cake requiring $\frac{1}{2}$ cupful of butter and 2 eggs?

11. A sponge cake can be made by using 6 eggs without baking powder or 3 eggs with 2 teaspoonfuls of baking powder. Find difference in cost when baking powder costs 45 cents per pound and 1 cupful weighs $5\frac{1}{2}$ ounces and eggs sell at above prices. What is the per cent of increase in cost?

12. One cupful of unsifted flour measures 1 cupful and $3\frac{1}{4}$ tablespoonfuls after sifting. Find excess of flour used in a recipe requiring 6 cupfuls, if flour is not sifted before using.

13. Sugar is 6 cents per pound; bread, 5 cents per loaf ($14\frac{1}{2}$ ounces); butter, 30 cents per pound; eggs, 25 cents per dozen (1 egg weighs 2 ounces); milk $6\frac{1}{4}$ cents per quart (1 cupful weighs $8\frac{7}{8}$ ounces). Express graphically the quantities of these foods that can be purchased for 25 cents.

14. Lard costs 16 cents a pound (1 pound measures 2 cupfuls). Suet costs 8 cents per pound. When tried out, 1 pound suet makes $1\frac{3}{4}$ cupfuls. Find the difference in cost of 3 pounds of lard and the same quantity of lard and suet, there being twice as much lard as tried-out suet in the mixed fat.

15. It is considered that less fat is absorbed by cooking in deep fat than in sautéing. Verify this statement. It took 1 tablespoonful of fat to sauté a croquette and $\frac{1}{2}$ tablespoonful of fat to sauté a fishball. In frying in deep fat 5 cupfuls were used for 45 croquettes and 90 fishballs.

Problems on Economy in Marketing

1. At market a quart can of tomatoes costs 10 cents and a peck of fresh tomatoes costs 15 cents. From the latter 7 pint cans can be made. Scalloped tomatoes can be made with canned or fresh fruit. Which is the cheaper to use? If 1 quart of tomatoes is used, how much cheaper is it?

Solution: 7 pints made from 1 peck cost 15 ¢.
1 pint cost $2\frac{1}{7}$ ¢.
1 quart cost $4\frac{2}{7}$ ¢.
1 quart canned cost 10 ¢.
Fresh tomatoes are 10 ¢ − $4\frac{2}{7}$ ¢, or $5\frac{5}{7}$ ¢ a quart cheaper.

2. A family uses 2 dozen cans of tomatoes during the winter. Find out how much is saved by canning the tomatoes at home when fresh tomatoes sell for 50 ¢ a bushel.

3. A can of corn costs 12 cents at market; it measures $2\frac{1}{3}$ cupfuls. Green corn costs 15 cents a dozen ears; 2 ears of corn make $1\frac{1}{8}$ cupfuls of stewed corn. Find the difference in cost of 1 can of corn and the same quantity of fresh corn.

4. 95 % of coffee sugar is pure sugar, while granulated sugar is 100 % pure. Coffee sugar costs $5\frac{1}{2}$ cents per pound, and granulated sugar costs 6 cents per pound. Which is the cheaper?

5. Granulated sugar sells at 6 cents a pound or $1.45 for 25 pounds. If a family uses 4 pounds of sugar a week, how much can be saved in a year by buying it in twenty-five pound quantities?

6. There are 2 cupfuls of granulated sugar to a pound, and $3\frac{1}{3}$ cupfuls of pulverized sugar to a pound. Find the equivalent in weight in pulverized sugar of $\frac{1}{2}$ cupful granulated sugar.

7. Both granulated and pulverized sugar are pure. The price of granulated sugar is 6 cents per pound and pulverized is 8 cents per pound. If the pulverized sugar is substituted for granulated, using the same quantity by measure, which is the cheaper to use? How much cheaper?

8. If a piece of meat having no waste sells for 9 cents per pound, and one which is one half bone sells for 8 cents per pound, which is the cheaper meat? How much cheaper is it?

9. How many eggs are used in your home each week? If eggs cost 28 cents per dozen in summer and 50 cents in winter, find the difference in the cost of eggs in your home during a summer and a winter month.

10. At the above prices, how many eggs can be purchased in the summer for the price of 1 dozen eggs in the winter?

11. 11% of egg is refuse. When eggs cost 40 cents per dozen, how much is spent for waste material?

12. 8 is what per cent of 12? 8 eggs weigh 1 pound. What is the cost of 1 pound of eggs when eggs sell for 35 cents per dozen?

13. One pound of roast beef contains twice as much nutrition as 1 pound of eggs. How many dozen eggs will furnish as much nutrition as 1 pound of beef?

14. Find the difference in cost when beef is 18 cents per pound and eggs are 40 cents per dozen.

15. If $\frac{1}{4}$ pound of roast beef is served to one person at a meal, how many eggs will it take to furnish as much nutrition?

16. One pound of ham had 2 ounces of fat and $\frac{3}{4}$ ounce of bone. What per cent of the meal was waste material?

17. When ham sells at 20 cents per pound how much is paid for waste?

18. One pork rib chop weighed 4 ounces. 1 ounce of this weight was bone. How much waste was there in 1 pound of pork chops?

19. Pork rib chops sold for 18 ¢ per pound. How much was paid for bone?

20. Two lamb rib chops weighed 6 ounces, 2 ounces of this weight being bone. What is the weight of the bone in 1 pound lamb rib chops? In $1\frac{1}{2}$ pounds chops?

21. A veal cutlet weighed 1 pound and 3 ounces. The bone and tough skin weighed 3 ounces. What per cent was waste?

22. If the cutlet cost 25 cents per pound, what was the cost of waste material per pound?

23. A veal chop weighed 4 ounces; the bone in it weighed 1 ounce. What per cent was waste?

24. If the chops cost 20 cents per pound, what was the cost of waste material per pound?

25. Which was the cheaper to buy, veal chops or cutlets?

26. A round steak weighed $2\frac{1}{4}$ pounds. The bone in it weighed 1 ounce. What per cent of the steak was bone?

27. Round steak sold at 18 cents per pound. What was the cost of waste from a pound?

28. Four pounds of shank contained 2 pounds bone. What per cent of the shank was bone?

29. Shank sold at $8\frac{1}{2}$ cents per pound. What was the cost of waste from $1\frac{1}{2}$ pounds?

30. If no use was made of the bone, which was the cheaper meat to buy, round or shank?

31. For biscuits, 2 level teaspoonfuls of baking powder are used with each cupful of flour. How many teaspoonfuls of baking powder does it take for 1 quart of flour? For $3\frac{1}{2}$ cupfuls of flour?

32. How much flour would $3\frac{1}{2}$ teaspoonfuls of baking powder leaven? $6\frac{3}{4}$ teaspoonfuls?

Problems on Economy of Labor and Fuel

1. Counting fuel and cost of materials, 4 dozen cookies can be made for 31 cents. At the bakery cookies cost 10 cents per dozen. Find the difference in cost of "home-made" and "bakery" cookies.

2. To mix and bake 4 dozen cookies at home requires 2 hours. If a woman can earn 15 cents per hour, is it cheaper for her to buy or make cookies? How much cheaper?

3. Meats prepared as stews or pot roasts should be cooked at simmering rather than boiling temperature. It has been found that the double gas burner of a range consumes 1 cubic foot of gas in 4 minutes, if it is wide open. 1 cubic foot of gas is consumed in 16 minutes 30 seconds by the same burner at simmering height. How much is saved in fuel in a month by cooking three stews for 3 hours, 3 times a week, if the meat is cooked at simmering rather than boiling temperature?

4. If water boils gently, 1 cubic foot of gas is consumed in 7 minutes; if it boils rapidly, 1 cubic foot is consumed in 4 minutes. If potatoes are cooked once a day, 25 minutes being required to cook them, how much is saved in the monthly gas bill, by boiling them gently rather than rapidly?

5. Old beets require 3 hours' cooking; turnips, 45 minutes'; and beans, $1\frac{1}{2}$ hours'. If these vegetables are each cooked once a week, how much fuel is saved a month by cooking them in *gently boiling* water?

6. It has been found that a small oven placed over one burner can be heated to a temperature of 383° F. by consuming 1 cubic foot of gas in 15 minutes. To heat the oven of a range to the same temperature, 1 cubic foot of gas is consumed in 3 minutes.

If a family used 2 roasts of beef per week, requiring $1\frac{1}{2}$ hours baking at 383° F., how much would be saved in a year by using the smaller oven? If the smaller oven costs $2.50, would it be economy to buy it for a year's use?

7. A roast fills the small oven, also 4 loaves of bread fill it. The larger oven holds 4 loaves of bread along with the roast. Bread requires one hour to bake. If a family uses 8 loaves of bread and 2 roasts of beef per week, what is the difference in the quantity of fuel consumed by the two ovens?

8. Rice in bulk costs 10 cents per pound; puffed rice in package costs 27 cents per pound. It takes 45 minutes to steam $\frac{1}{2}$ pound of rice over a burner consuming 1 cubic foot of gas in 7 minutes. What is the difference in cost of the rice cooked at home and that purchased already prepared?

9. Corn meal sells at $2\frac{1}{2}$ cents per pound, corn flakes 14.3 cents per pound. One pound of corn meal requires 2 hours' cooking, the burner consuming 1 cubic foot of gas in 16 minutes 30 seconds. What is the difference in cost per pound of the two kinds of corn meal? If the fireless cooker is used, requiring 15 minutes' cooking over the burner for one pound of corn meal, what is the difference in price per pound, the burner for the fireless cooker consuming 1 cubic foot of gas in 7 minutes?

10. Pea soup to serve six can be prepared from 1 can peas, costing 12 cents, or from 1 cupful of split peas costing 6 cents per pound (1 pound split peas measures 2 cupfuls). 15 minutes are required to cook the canned peas and $3\frac{1}{2}$ hours to cook the split peas. The simmering burner on which these foods should be cooked consumes 3.6 cubic feet per hour; gas costs 75 cents per 1000 cubic feet. Which kind of peas is it cheaper to use?

11. Round steak costs 16 cents per pound, and porterhouse 22 cents per pound. To make round steak palatable requires 2 hours to cook it, and the porterhouse requires 5 minutes. Porterhouse should be cooked on the large burner, burning 8.6 cubic feet per hour; round steak should be cooked for 10 minutes on the same burner and for the remainder of the time on the simmering burner, burning 3.6 cubic feet per hour. Find the difference in cost.

Problems on Formation of Recipes

Proportions of flour and baking powder for quick breads:

2 teaspoonfuls of baking powder to 1 cupful of flour, when no eggs are used.

$1\frac{1}{2}$ teaspoonfuls of baking powder to 1 cupful of flour when 1 egg is used with 3 cupfuls of flour.

1 teaspoonful of baking powder to one cupful of flour when 2 or more eggs are used with 2 or 3 cupfuls of flour.

The proportion of flour and salt is:

$\frac{1}{4}$ teaspoonful of salt to 1 cupful of flour.

The proportion of flour and fat is:

1 tablespoonful of fat to 1 cupful of flour.

The proportion of flour and moisture for batters and doughs:

Pour batter — 1 part liquid with 1 to $1\frac{1}{2}$ parts of flour.
Drop batter — 1 part liquid with 2 to $2\frac{1}{2}$ parts of flour.
Soft dough — 1 part liquid with $2\frac{2}{3}$ parts of flour.
Stiff dough — 1 part liquid with 3 or more parts of flour.

Mixtures for griddle cakes and waffles are pour batters.

For griddle cakes equal parts of flour and liquid should be used; for waffles, 1 part liquid with $1\frac{1}{3}$ parts of flour.

Write recipes for:

1. Griddle cakes using 2 cupfuls of flour as the basis and 1 egg.

2. Waffles using the same quantities of given ingredients *in 1.*

3. Griddle cakes using $2\frac{1}{2}$ cupfuls of flour as the basis and 2 eggs.

4. Griddle cakes using $3\frac{3}{4}$ cupfuls of flour as the basis with 2 eggs.

5. Waffles using the same quantities of given ingredients in 4.

6. Eggs contain 10.7 % fat; 1 egg weighs 2 ounces. Butter contains 85 % fat; 2 cupfuls weigh 1 pound. Oleomargarine contains 83 % fat; 2 cupfuls weigh 1 pound. When butter sells for 40 cents per pound and eggs for 50 cents per dozen, which is the cheaper source of fat?

7. When oleomargarine sells for 25 cents per pound, which is the cheaper source of fat, oleomargarine or egg?

Write recipes for:

8. Griddle cakes, making them as economically as possible, when eggs sell for 50 cents per dozen, using 3 cupfuls of flour as the basis.

9. Waffles, making them as economically as possible, when eggs sell for 50 cents per dozen, using $2\frac{3}{4}$ cupfuls of flour as the basis.

10. Griddle cakes, using 4 cupfuls of flour and 1 egg, and adding enough oleomargarine (besides the usual quantity of fat) to furnish as much fat as two more eggs.

11. Waffles, using 3 cupfuls of flour and 2 eggs, and adding enough butter (besides the usual quantity) to furnish as much fat as one more egg.

12. Find the per cent saved if oleomargarine were substituted for butter in 11. If a family uses griddle cakes made from recipe 1 for five mornings per week, find per cent saved during a month if oleomargarine is used instead of butter.

13. Find difference in cost of griddle cakes during a month made from recipe 1 for three mornings per week when eggs sell at 28 cents per dozen and butter at 30 cents per pound, and when eggs sell at 50 cents per dozen and butter at 40 cents per pound.

Mixtures for "drop" biscuits are drop batters, using 1 part liquid with 2 parts flour. For the ordinary molded biscuits the mixture is a soft dough. Shortcakes may be made using a drop biscuit mixture, increasing the fat two or three times. These mixtures contain no eggs.

Write recipes for:

14. Drop biscuits, using 2 cupfuls of flour as a basis.

15. Drop biscuits, using $2\frac{3}{4}$ cupfuls of flour as a basis.

16. Molded biscuits, using $3\frac{1}{2}$ cupfuls of flour as a basis.

17. Molded biscuits, using $2\frac{1}{3}$ cupfuls of flour as a basis.

18. From $\frac{1}{3}$ of a cupful of flour, 2 drop biscuits can be made. Write a recipe containing enough material to make $1\frac{1}{2}$ dozen biscuits.

19. From 2 cupfuls flour, 1 dozen small molded biscuits can be made. Write a recipe containing enough material for 40 biscuits.

When sour milk and baking soda are used to leaven a quick bread, $\frac{1}{2}$ teaspoonful of soda is used with 1 cupful of sour milk. These ingredients are always combined in this proportion with no reference to the number of eggs used, since a definite quantity of alkali must be used to neutralize a certain quantity of acid. Baking powder contains both alkali and acid properly proportioned, hence complete neutralization takes place, no matter what quantity of baking powder is used.

Write recipes for:

20. Griddle cakes, using 1 pint of sour milk as a basis and 1 egg.

21. Griddle cakes, using $3\frac{1}{2}$ cupfuls of sour milk as the basis and 2 eggs.

22. *Waffles, using* $2\frac{1}{2}$ cupfuls sour milk as a basis and 1 egg.

23. Waffles, using 4 cupfuls of sour milk as a basis and 2 eggs.

24. From $\frac{1}{2}$ cupful of flour, enough batter can be made to fill the waffle irons. Write a recipe containing enough material to fill the waffle iron five times, making it as economically as possible.

25. One fourth cupful of flour makes 4 griddle cakes. Write a recipe containing enough material for 6 people, allowing 5 cakes for each person.

Problems on Nutritive Value of Foods

1. According to Fisher's "100 calorie portion," $\frac{5}{8}$ cupful of milk yields 100 great calories. How many great calories are produced by 1 quart of milk?

2. One large egg, 5 teaspoonfuls of sugar, and $\frac{5}{8}$ cupful of milk each yield 100 great calories. For soft custards, 1 egg and $\frac{1}{8}$ cupful of sugar are used with 1 cupful of milk. Find food value of 1 pint of custard.

3. One large egg and one shredded wheat biscuit each yield 100 great calories. The dietary standard for a girl of twelve years is 1276.8 great calories. The food of breakfast should produce $\frac{1}{3}$ of this quantity. What part of an ideal breakfast's nutrition is produced by an egg and a biscuit?

4. One pint oysters weighs $1\frac{5}{8}$ pounds. Oysters contain 6 % protein, 1.3 % fat, and 3.3 % carbohydrates. One pint of milk weighs 1 pound. Milk contains 3.3 % protein, 4 % fat, and 5 % carbohydrates. One ounce protein and carbohydrates each yields 116 great calories; one ounce fat, 264 great calories. Find difference in the nutritive value of milk and oysters.

5. For oyster stew to one quart of oysters are added 1 quart of milk and 4 tablespoonfuls butter. Butter measures 2 cupfuls to the pound; it contains 0.1 % protein, 85 % fat, and 0.0 % carbohydrates. What is the food value of oyster stew?

6. The given quantity of oyster stew serves eight. If a woman of average size requires enough food per day to produce 2090 great calories, and the food of luncheon should supply one fourth of the day's ration, what part of the luncheon ration will one portion of oyster stew furnish?

7. Calculate the number of great calories produced by scalloped potatoes, made by using 4 potatoes, 1 cupful of milk, 2 tablespoonfuls of butter, and 2 tablespoonfuls of flour. The edible portion of potatoes weighs 22 ounces. They contain 2.2 % protein, 0.1 % fat, and 18.4 % carbohydrates.

8. One pint of milk weighs 1 pound; it contains, 3.3 % protein, 4 % fat, and 5 % carbohydrates. One cupful of butter weighs 8 ounces; it contains 1 % protein, 85 % fat, and 0.0 % carbohydrates. One cupful of flour weighs $\frac{1}{4}$ pound; it contains 11.4 % protein, 1 % fat, and 75.1 % carbohydrates. One ounce protein and carbohydrates each yields 116 great calories, and one ounce of fat 264 great calories. The quantity given above is sufficient to serve six. If a day's ration for a man of ordinary size should yield 2280 great calories and the food of dinner should yield $\frac{5}{12}$ of this quantity, what part of the nutritive value of the dinner is produced by the scalloped potatoes?

9. According to Hutchison, it has been calculated that the loss of starch from 1 bushel of potatoes which have been pared and soaked in water is equivalent to the nutrients contained in 1 pound of beef steak. The latter yields 745 great calories. If 1 ounce of starch yields 116 great calories, what part of a pound of starch is lost by paring and soaking the potatoes in water?

10. One bushel contains 144 potatoes. If six potatoes are used daily by a family, how many ounces of starch are lost *during the month* if the potatoes are pared and soaked?

11. Potatoes contain 19.1 % starch; one pared potato weighs 5.5 ounces. In the potatoes of the above problem, what per cent of the starch is lost?

12. What is the difference in the fuel value of steamed rice, in which 3 cupfuls of water are used and that in which one half the liquid is milk, milk containing 3.3 % protein, 4 % fat, and 5 % carbohydrates?

13. One banana weighs $3\frac{3}{4}$ ounces. Bananas contain 1.3 % protein, 0.6 % fat, and 22 % carbohydrates. One loaf of bread weighs $14\frac{1}{2}$ ounces. Bread contains 9.2 % protein, 1.3 % fat, and 53.1 % carbohydrates. If one loaf of bread is cut into 15 slices, how many slices of bread are equivalent to 1 banana in fuel value?

14. One can of peas weighs 1 pound $4\frac{3}{4}$ ounces; it contains 3.6 % protein, 0.2 % fat, and 9.8 % carbohydrates. One cupful dried peas weighs $7\frac{1}{4}$ ounces; it contains 24.6 % protein, 1 % fat, and 62 % carbohydrates. Find difference in fuel value.

15. There are $4\frac{3}{4}$ cupfuls rolled oats to the pound. One cupful of cereal, after cooking serves six. It contains 16.7 % protein, 7.3 % fat, and 66.2 % carbohydrates. Find the fuel value of enough rolled oats to serve 1 person.

16. There are $1\frac{7}{8}$ cupfuls of rice to the pound. Rice contains 8 % protein, 0.3 % fat, and 79 % carbohydrates. Find the food value of one cupful of rice.

17. Find the fuel value of one pound rolled oats, corn meal, rice, and macaroni. Corn meal contains 9.2 % protein, 1.9 % fat, and 75.4 % carbohydrates; macaroni contains 13.4 % protein, 0.9 % fat, and 74.1 % carbohydrates.

18. Compare these cereals graphically according to Fisher.

19. One loaf of bread contains three fourths as much nutrition as one pound of rice. Bread costs 5 cents per loaf and rice 8 cents per pound. Which is the cheaper?

20. Cooked eggs contain 12 % protein, and cooked roast beef 22.3 % protein. One egg weighs two ounces. How many eggs furnish as much protein as one pound roast beef?

21. A pound of round steak contains ten times as much nutrition as a pound of lettuce. How many pounds of lettuce would furnish as much nutrition as one pound of round steak?

22. Find the difference in the cost of one pound of round steak and the quantity of lettuce that would furnish the same nutrition when round steak is 16 cents per pound and lettuce 8 cents per pound.

23. The fuel value of one pound of sugar is 1860 great calories; bread, 1215; butter, 3605; eggs, 635; and milk, 325. Express graphically these food values.

24. Express graphically the food values of the quantities of the above foods that can be purchased for 25 cents.

LIST OF CONSTRUCTIONS

LIST OF THEOREMS

Lines

Triangles

INDEX